Article:
Second Order Systems

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Preface

This article gives an introduction to second order dynamic systems, and it can be downloaded for free from http://techteach.no. It can be used as supplementary material to the book *Dynamics and Control*, which can be purchased from http://techteach.no.

1 Transfer function model

A standard second order transfer function model (with u as input variable and y as output variable) is

$$y(s) = \frac{K{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}u(s) \equiv \frac{K}{(\frac{s}{\omega_0})^2 + 2\zeta\frac{s}{\omega_0} + 1}u(s) \tag{1}$$

where K is the *gain*, ζ (zeta) [dimension 1] is the *relative damping factor,* and ω_0 [rad/s] is the *undamped resonance frequency.*

Eksempel 1 ***Second order system: Mass-spring-damper***

Figure 1 shows a mass-spring-damper-system. (One concrete example is the wheel suspension system on a car.) We will assume that the spring force is zero when y is zero. Force balance yields

$$\begin{aligned} m\ddot{y}(t) &= F(t) - F_d(t) - F_f(t) \\ &= F(t) - D\dot{y}(t) - K_f y(t) \end{aligned} \tag{2}$$

Taking the Laplace transform (while assuming that the initial values of the y and $\dot{y}$ are zero) yields

$$ms^2y(s) = F(s) - Dsy(s) - K_f y(s) \tag{3}$$

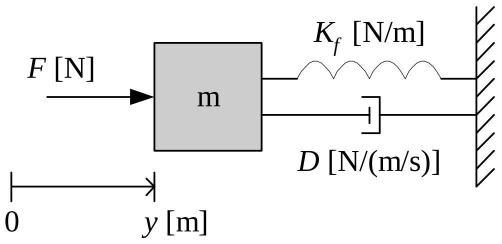


Figure 1: Mass-spring-damper

Solving for $y(s)$:

$$y(s) = \underbrace{\frac{1}{ms^2 + Ds + K_f}}_{H(s)} F(s) \tag{4}$$

So, the transfer function from force F to position y is

$$H(s) = \frac{1}{ms^2 + Ds + K_f} \tag{5}$$

To find the standard parameters of this second order transfer function, we must transform the transfer function to one of the equivalent standard forms given by (1). Let us here choose the first one:

$$H(s) = \frac{\overbrace{\frac{1}{m}}^{K\omega_0{}^2}}{\underbrace{s^2 + \frac{D}{m}s + \frac{K_f}{m}}_{s^2+2\zeta\omega_0 s+\omega_0{}^2}} \tag{6}$$

By equating the coefficients and using the following parameters values: $m = 20$ kg, $D = 4$ N/(m/s) and $K_f = 2$ N/m, we get

$$K = \frac{1}{K_f} = 0.5 \text{ [m/N]} \tag{7}$$

$$\omega_0 = \sqrt{\frac{K_f}{m}} = \sqrt{0,1} = 0.32 \text{ [rad/s]} \tag{8}$$

$$\zeta = \frac{D}{2\sqrt{mK_f}} = 0.32 \tag{9}$$

[End of Example 1]

2 Classification of second order systems

2.1 Overview

We will classify second order systems from the shape of the step response. We assume that the input variable $u(t)$ is a step of amplitude U, which Laplace transformed is $u(s) = U/s$. Then the Laplace transformed time-response becomes

$$y(s) = H(s)u(s) = \frac{K{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}\frac{U}{s} \tag{10}$$

The shape of the time-response $y(t)$, which is calculated as the inverse Laplace transform of $y(s)$, depends on the poles. The poles are the roots of the characteristic equation $a(s)$:

$$a(s) = s^2 + 2\zeta\omega_0 s + {\omega_0}^2 = 0 \tag{11}$$

The poles p_1 and p_2 are the roots of $a(s)$ and becomes

$$p_1,\ p_2 = -\zeta\omega_0 \pm \sqrt{\zeta^2 - 1}\ \omega_0 \tag{12}$$

The value of ζ determines whether the poles are real or complex conjugate:

- If $\zeta \geqslant 1$, the poles are real and given by (12).
- If $0 \leq \zeta < 1$, the poles are complex conjugate:

$$p_1,\ p_2 = \underbrace{-\zeta\omega_0}_{\text{Re}} \pm j\underbrace{\sqrt{1 - \zeta^2}\ \omega_0}_{\text{Im}} \tag{13}$$

  Figure 2 shows the pole placement when the poles are complex conjugate.

Figure 3 classifies second order systems by the value of ζ. (This is a common way to do the classification.) The step responses referenced in the figure can be calculated by taking the inverse Laplace transform of (10), but the detailed calculations are not shown here.

In the following are given simulated step responses and pole plots for representative examples of overdamped, underdamped, and undamped systems. The parameter values are shown on the front panels of the simulators in the respective figures.

In all cases the steady-state value of the step response is

$$y_s = KU \tag{14}$$

because the static transfer function is K.

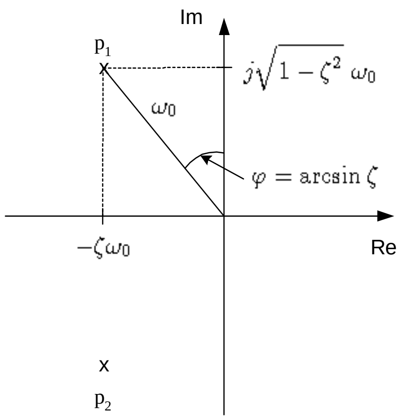


Figure 2: Pole placement for second order systems when the poles are complex conjugate. The poles are given by (13).

2.1.1 Overdamped systems

Figure 4 shows the step response and the poles for an example of an overdamped system.

Comments:

- The step response has no overshoot.
- The poles p_1 and p_2 are real and distinct:

$$p_1,\ p_2 = -\zeta\omega_0 \pm \sqrt{\zeta^2 - 1}\ \omega_0 \tag{15}$$

  The transfer function can therefore be written on the form

$$H(s) = \frac{Kp_1p_2}{(s-p_1)(s-p_2)} = \frac{K}{(T_1s+1)(T_2s+1)} \tag{16}$$

  This implies that the second order system can be split into two first order subsystems having time-constants T_1 and T_2, respectively. The largest of these time-constants can be denoted the *dominating time-constant.*

2.1.2 Underdamped system

Figure 5 shows the step response and the poles for an example of an underdamped system.

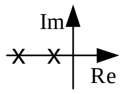
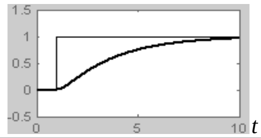
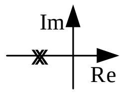
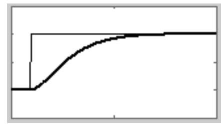
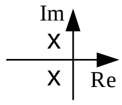
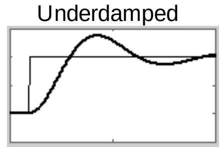
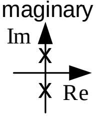
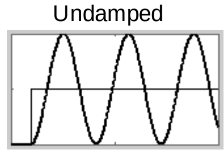
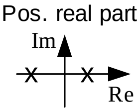
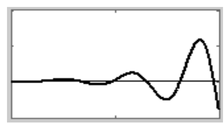
Value of ζ	Poles p_1 and p_2	Type of step response $y(t)$
$\zeta > 1$	Real and distinct 	Overdamped 
$\zeta = 1$	Real and multiple 	Critically damped 
$0 < \zeta < 1$	Complex conj. 	Underdamped 
$\zeta = 0$	Imaginary 	Undamped 
$\zeta < 0$	Pos. real part 	Unstable 

Figure 3: Classification of second order systems by the value of ζ

Comments:

- The poles are complex conjugate:

$$p_1,\ p_2 = -\zeta\omega_0 \pm j\sqrt{1-\zeta^2}\ \omega_0 \tag{17}$$

- The less ζ, the less damping in the step response. It can be shown that the less ζ, the more dominating imaginary part over the real part of the poles. This is a general property of poles: The larger imaginary part relative to the real part, the less damping in the time-response. Figure 6 shows the step-response for various values of ζ.

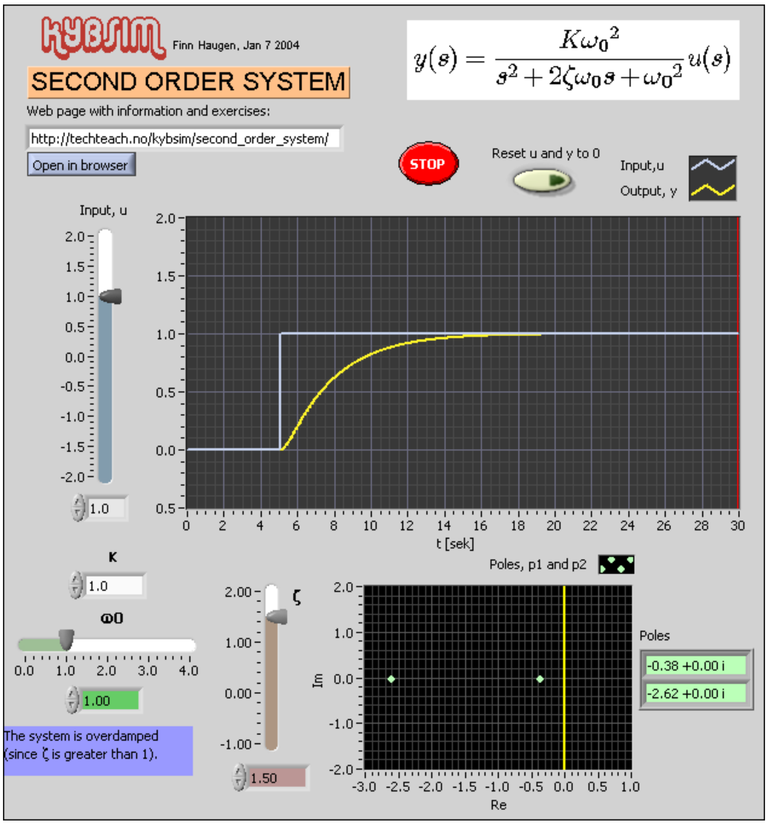


Figure 4: Step response and the poles for an example of an overdamped system

- The *overshoot factor* δ of the step response is defined as

$$\delta = \frac{y_{\max} - y_s}{y_s} \tag{18}$$

where y_s is the steady-state value of the step-response. It can be shown that δ is a function of the relative damping factor ζ, as follows:

$$\delta = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \tag{19}$$

δ is plotted as a function of ζ in Figure 7.

The inverse function of (19) is

$$\zeta = \frac{|\ln \delta|}{\sqrt{\pi^2 + (\ln \delta)^2}} \tag{20}$$

A few examples: Overshoot $\delta = 0.1$, that is, 10% overshoot, corresponds to $\zeta = 0.6$. Overshoot $\delta = 0$ (zero overshoot) corresponds to $\zeta = 1$ (critically damped system).

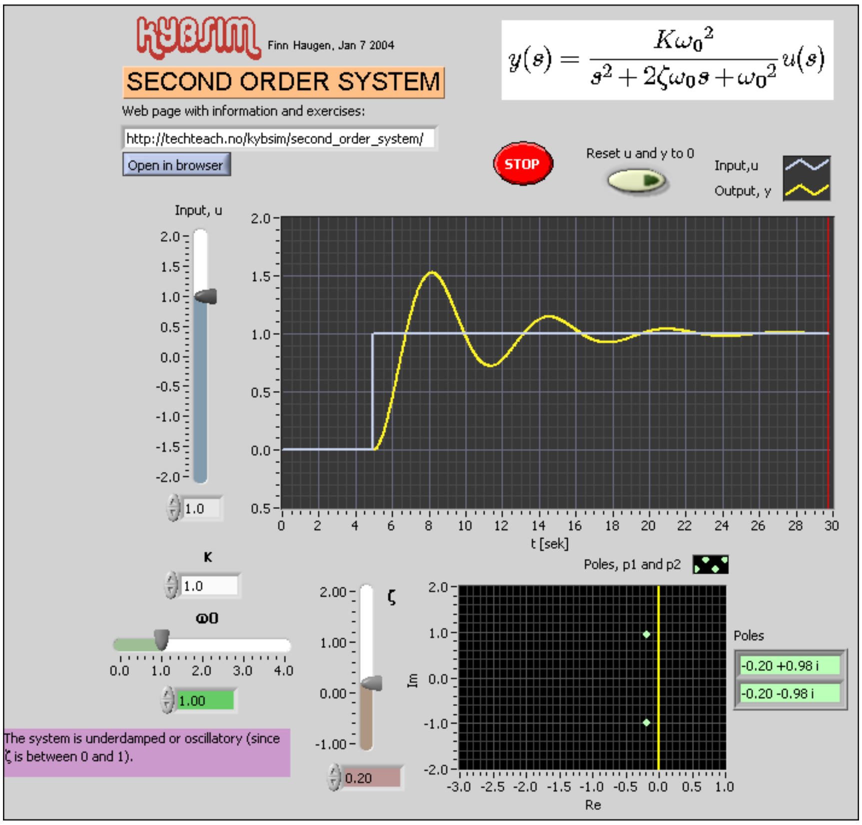


Figure 5: Step response and poles for an example of an underdamped system

- Simulations shows that the 63% response-time of the step response is approximately

$$T_r \approx \frac{1.5}{\omega_0} \tag{21}$$

  ω_0 expresses in a way how quick the system is. ω_0 is the distance from origin to the poles, see Figure 2. This is a general property of poles: The longer distance from origin, the faster the dynamics of the system.

- It can be shown that the frequency in the oscillations are

$$\beta = \sqrt{1-\zeta^2}\,\omega_0 \text{ [rad/s]} \tag{22}$$

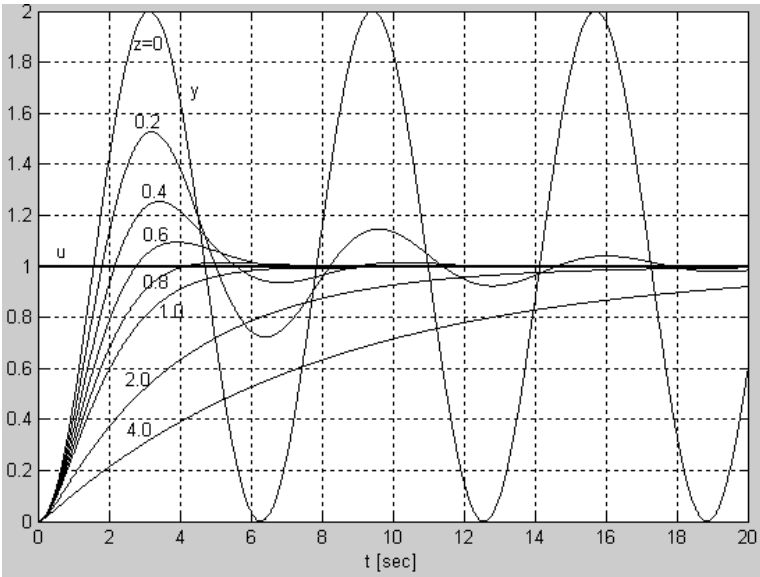


Figure 6: Step-response for various values of ζ for second order systems

2.1.3 Undamped system

Figure 8 shows the step response and the poles for an example of an undamped system.

Comments:

- The step response is undamped, steady-state oscillations:

$$y(t) = KU(1 - \cos\omega_0 t) \tag{23}$$

  The frequency of the oscillations in rad/s is ω_0 — therefore the name *undamped resonance frequency*.

- The poles are purely imaginary:

$$p_1,\ p_2 = \pm j\omega_0 \tag{24}$$

  The real part is zero, which is an explanation of why the step response is undamped. In general, damping is due non-zero real part of poles.

Eksempel 2 ***Control system***

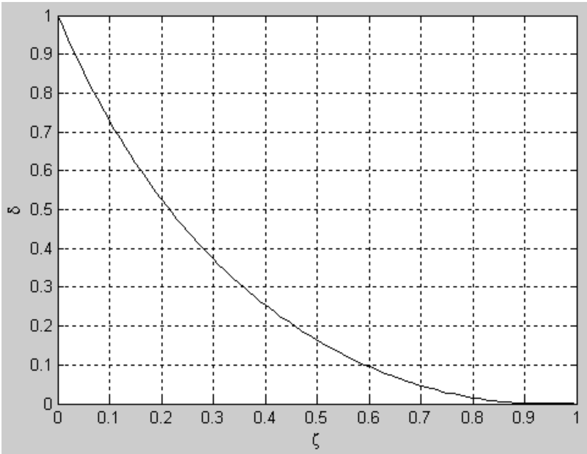


Figure 7: Overshoot factor δ plotted as a function of the relative damping factor ζ, cf. (19)

Figure 9 shows a control system for the angular position of an electro-motor. Figure 10 shows a transfer function based block diagram for the control system. We assume that the individual transfer functions with parameter values are as follows:

$$H_u(s) = \frac{K_u}{(Ts+1)\,s} = \frac{1}{(s+1)\,s} \tag{25}$$

$$H_d(s) = \frac{K_d}{(Ts+1)\,s} = \frac{-1}{(s+1)\,s} \tag{26}$$

$$H_c(s) = K_p \quad \text{(proportional controller)} \tag{27}$$

$$H_s(s) = K_m = 1 \tag{28}$$

$$H_{sm}(s) = K_{sm} = 1 \tag{29}$$

(Some information about the background of $H_u(s)$: The transfer function from the manipulating or controlling variable u to the speed v is $K_1/(Ts+1)$, and the transfer function from v to position y is $1/s$ which is an integrator. The process to be controlled is thus a “first order system in series with an integrator”.)

Let us calculate the controller gain K_p from specifications for the transfer function $H_{y_r,y}(s)$ from the position reference y_r to the position y (the specifications are presented soon). First, we must find $H_{y_r,y}(s)$. From the

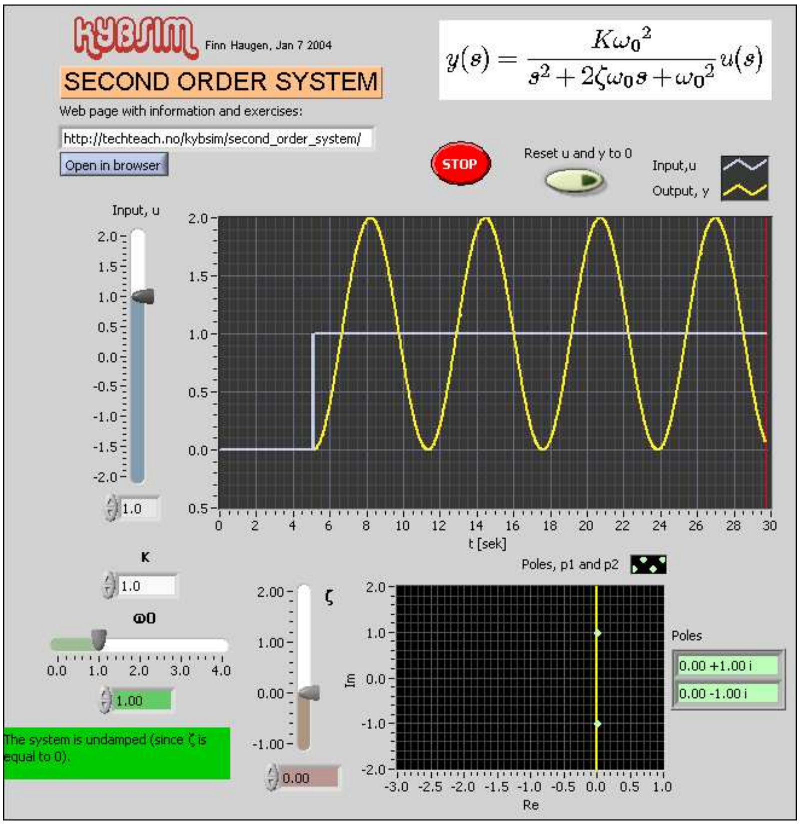


Figure 8: Step response and the poles for an example of an undamped system

block diagram in Figure 10 we will find

$$H_{y_r,y}(s) = \frac{H_{sm}(s)H_c(s)H_u(s)}{1 + H_s(s)H_c(s)H_u(s)} \tag{30}$$

$$= \frac{K_{sm}K_p\frac{K_u}{(Ts+1)s}}{1 + K_sK_p\frac{K_u}{(Ts+1)s}} = \frac{K_p\frac{1}{(s+1)s}}{1 + K_p\frac{1}{(s+1)s}} \tag{31}$$

$$= \frac{K_p}{s^2 + s + K_p} \tag{32}$$

$$= \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \tag{33}$$

By equating coefficients between (32) and (33) we get:

$$K = 1 \tag{34}$$

$$\omega_0 = \sqrt{K_p} \tag{35}$$

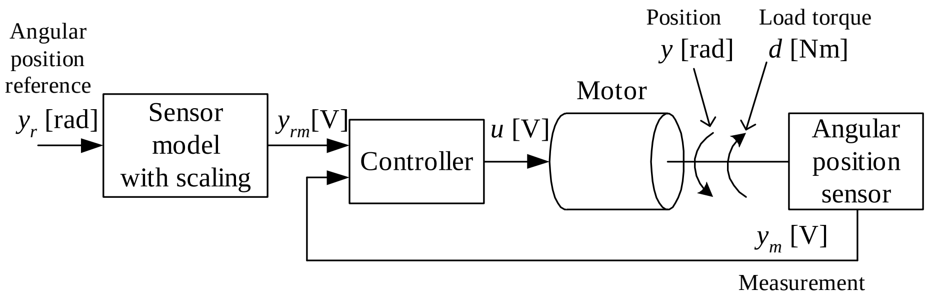


Figure 9: Feedback control system for the angular position of an electro-motor

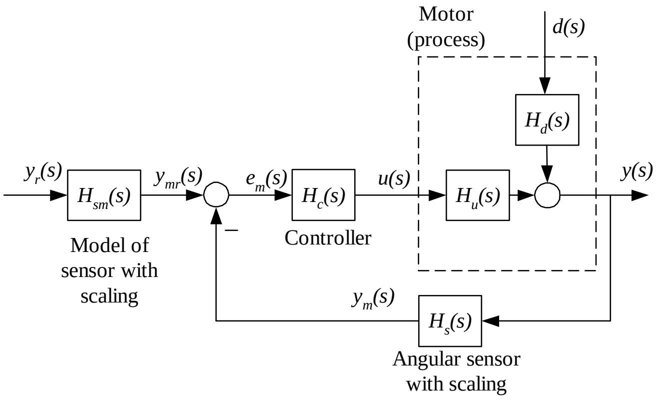


Figure 10: Transfer function based block diagram for the positional control system shown in Figure 9

$$\zeta = \frac{1}{2\sqrt{K_p}} \tag{36}$$

From (35) and (36) we can calculate K_p either from a specified ω_0, which expresses the quickness of the system since the 63% response-time is $T_r \approx 1.5/\omega_0$, cf. (21), or from a specified ζ, which expresses the damping of the system. Good stability is the most important property of a control system, so we use (36) as the basis for calculation of K_p. Let us say that $\zeta = 0.6$ is a reasonable value. $\zeta = 0.6$ gives 10% overshoot ($\delta = 0.1$) in the step response, cf. (20). From (36) we get

$$K_p = \frac{1}{4\zeta^2} = \frac{1}{4 \cdot 0.6^2} = 0.69 \tag{37}$$

The response-time of the control system then becomes

$$T_r \approx \frac{1.5}{\omega_0} = \frac{1.5}{\sqrt{K_p}} = \frac{1.5}{\sqrt{0.69}} = 1.8 \text{ sec} \tag{38}$$

Figure 11 shows simulated step responses in the position y (it is a step of

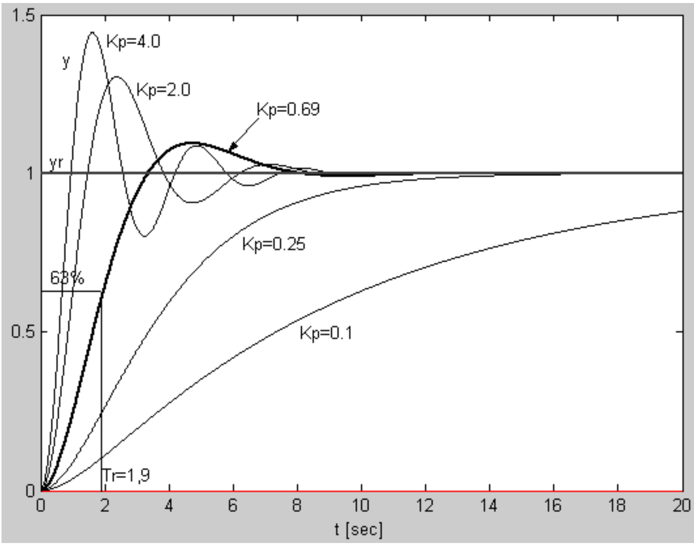


Figure 11: Simulated step responses in the angular position y (it is a step of amplitude 1 in the reference r) for several K_p-values

amplitude 1 in the reference r) for the calculated $K_p = 0.69$ and for several other K_p-values. We see that the step response is quicker and less damped the larger the K_p-value. (This is a typical consequence of increasing the controller gain in control systems.)

From Figure 11 we read off a response-time of $T_r = 1.9$ (for $K_p = 0.69$), so (38) is quite accurate.

[End of Example 2]