Exercises to Advanced Dynamics and Control

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Contents

Preface

This book contains exercises with soutions to *Advnaced Dynamics and Control*, TechTeach, August 2010. The exercises can all be solved with just manual calculations (using paper and pencil). So, computer-based exercises are not covered.[1]

The following freely available material may also be useful on exercises:

- **SimView**, which is a collection of ready-to-run simulators.
- **TechVids**, which is a collection of instructional streaming videos. In the videos theory is explained, and simulators are run and explained. You can download these simulators, and run them while you are playing the videos.

This book is organized in chapters and sections which correspond to the text-book.

Appendix A describes mathematical models of a some physical systems. Several exercises in this book are based on these systems. Their models can be used in computer-based exercises with e.g. Matlab/Simulink or LabVIEW.

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[1] The formulation of computer-based exercises depends largely on the tool being used, e.g. MATLAB/SIMULINK, LabVIEW, Scilab/Scicos, Octave., and the tool being used may vary from one school/university to another. Therefore, this book does not contain such exercises.

Part I

EXERCISES

Chapter 1

State-space models

1.1 Introduction

No exercises here.

1.2 A general state-space model

Exercise 1.1

Figure 1.1 shows two coupled liquid tanks. u_1 and u_2 are control signals. Mass balance of tank 1 is

$$\rho A_1 \dot{h}_1 = \rho \underbrace{K_p u_1}_{q_1} - \rho \underbrace{K_{v_1} \sqrt{\frac{\rho g h_1}{G}}}_{q_2} \tag{1.1}$$

Mass balance of tank 2 is

$$\rho A_2 \dot{h}_2 = \rho \underbrace{K_{v_1} \sqrt{\frac{\rho g h_1}{G}}}_{q2} - \rho \underbrace{K_{v_2} u_2 \sqrt{\frac{\rho g h_2}{G}}}_{q_3} \tag{1.2}$$

Valve 1 has fixed opening. Valve 2 is a control valve with control signal u between 0 and 1. The square root functions stems from the common valve characteristic which expresses that the flow is proportional to the square root of the pressures drop across the valve. Here, the pressure drops are assumed to be equal to the hydrostatic pressures at the bottom the tanks.

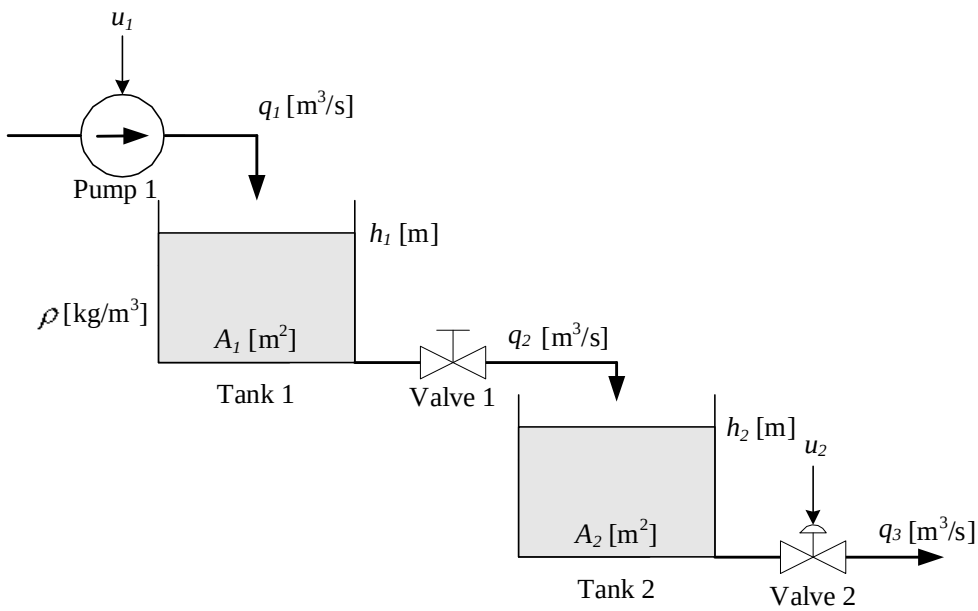

Figure 1.1:

For example, for tank 1 the hydrostatic pressure is $\rho g h_1$. The parameter G is the relative density of the liquid.[1]

Assume that the input variables are u_1 and u_2, and that the output variables are $y_1 = h_1$ and $y_2 = h_2$. Write the model (1.1) – (1.2) as a state-space model. Is the state-space model linear or nonlinear?

1.3 Linear state-space models

Exercise 1.2

Write the following model as a state-space model on matrix-vector form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ 2\dot{x}_2 &= 8u_2 - 6x_2 - 2x_1 + 4u_1 \\ y &= 5x_1 + 7u_1 + 6x_2 \end{aligned} \tag{1.3}$$

[1] $G = \rho/\rho_{water}$.

1.4 Linearization of non-linear models

Exercise 1.3

Given the state-space model (23.2) – (23.4). Linearize the model.

Chapter 2

Frequency response

2.1 Introduction

No exercises here.

2.2 How to calculate frequency response from sinusoidal input and output

Exercise 2.1

Figure 2.1 shows the input signal and the corresponding output signal of a system.

1. What is the frequency of the signal in Hz and in rad/s?
2. Calculate the amplitude gain and the phase lag at the frequency found in Problem 1 above. What is the amplitude gain in dB?

Exercise 2.2

Figure 2.2 shows a Bode diagram of a system. Assume that the input signal u is a sinusoid of amplitude $U = 0.8$ and frequency $\omega = 1.0$ rad/s. Write the corresponding steady-state output response $y_s(t)$.

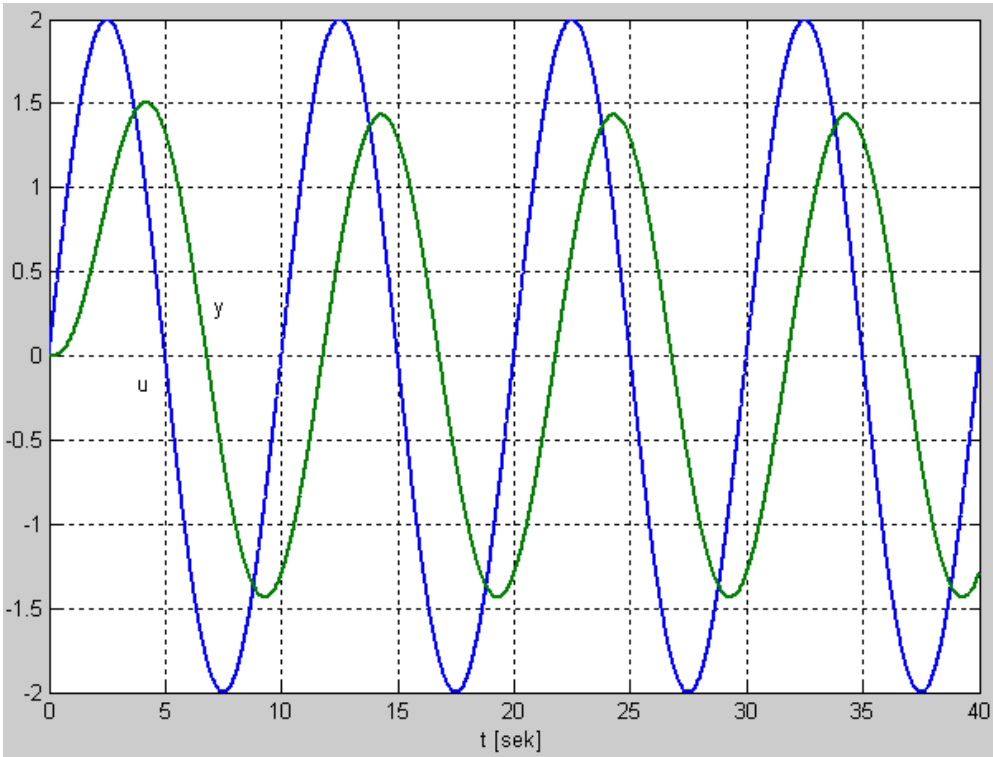

Figure 2.1:

2.3 How to calculate frequency response from transfer functions

Exercise 2.3

Calculate the frequency response functions $A(\omega)$ and $\phi(\omega)$ of the transfer function

$$H(s) = \frac{K}{(1+T_1 s)(1+T_2 s)} e^{-\tau s} \tag{2.1}$$

2.4 Application of frequency response: Signal filters

Exercise 2.4

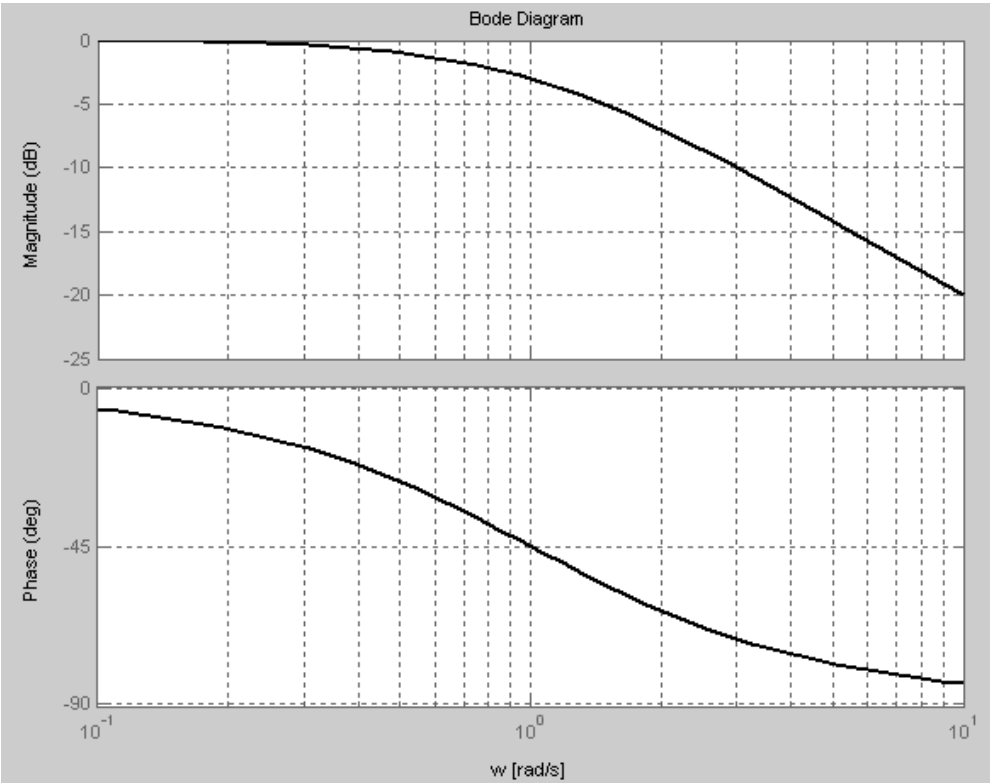

Figure 2.2:

Assume that it is specified that a given RC filter shall have bandwidth 100 Hz. Find proper values of the resistance R the capacitance C. (Tip: C can be selected between 10^{-4} and 10^{-6} F because this gives a practical size of the capacitor. Unless you insist on some other value, you can use 10^{-5} F.)

Chapter 3

Frequency response analysis of feedback control systems

3.1 Introduction

No exercises here.

3.2 Definition of setpoint tracking and disturbance compensation

Exercise 3.1

Figure 3.1 shows a principal block diagram of a control system. The setpoint y_{SP} and the disturbance v are two input signals to the control system. There is actually a third input that is present in practical control systems – more or less, namely measurement noise n, which is typically a random signal. Assume that n is added to the measurement signal that is the output of the sensor. Most control systems contains a lowpass filter which attenuates the measurement noise so that the resulting measurement signal entering the controller becomes smoother.

Include n and the filter in the block diagram shown in Figure 3.1 (make a new drawing).

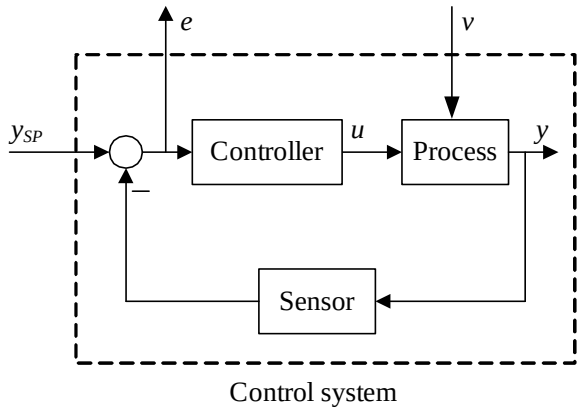

Figure 3.1:

3.3 Definition of characteristic transfer functions

Exercise 3.2

Assume given a control system as shown in Figure 3.2. The transfer functions are as follows:

$$H_u(s) = \frac{K_u}{T_u s + 1} e^{-\tau s} \tag{3.1}$$

$$H_v(s) = \frac{K_v}{T_v s + 1} e^{-\tau s} \tag{3.2}$$

$$H_m(s) = K_m \tag{3.3}$$

$$H_c(s) = K_p \frac{T_i s + 1}{T_i s} \text{ (PI controller)} \tag{3.4}$$

Find the loop transfer function $L(s)$, the sensitivity function $S(s)$, and the tracking function $T(s)$.

3.4 Frequency response analysis of setpoint tracking and disturbance compensation

Exercise 3.3

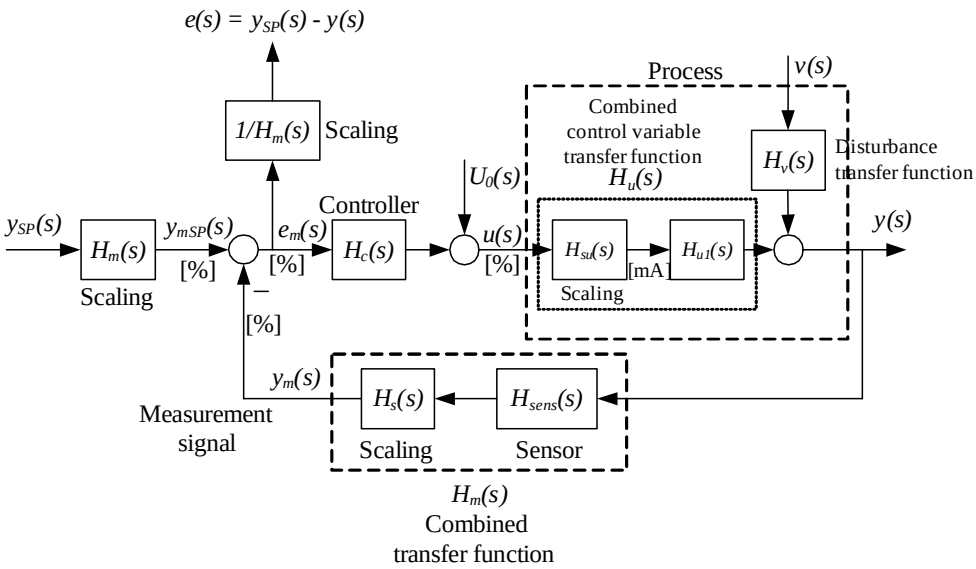

Figure 3.2:

Figure 3.3 shows the amplitude gain curves of the loop transfer function L, the tracking function T and the sensitivity function S of a feedback control system.

1. Read off from the frequency response curves the following three alternative bandwidths:

   - The crossover frequency ω_c.
   - The -3 dB frequency ω_t of the tracking function
   - The -11dB frequency ω_s of the sensitivity function

2. Assume that the setpoint is a sinusoid of amplitude $A_{SP} = 4$ and frequency 1 rad/s. What is the amplitude, A_y, of the steady-state sinusoidal process output variable? What is the amplitude, A_e, of the steady-state sinusoidal control error?

3. Assume that the process disturbance is a sinusoid of frequency 1 rad/s. Assume that this disturbance creates a steady-state sinusoidal response in the process output variable of amplitude $A_{y_{\mathrm{OL}}}0.5$ when the process is controlled with a constant control system, i.e. in open loop control. What is the amplitude, $A_{y_{\mathrm{CL}}}$, of the process output variable using feedback control, i.e. in closed loop control?

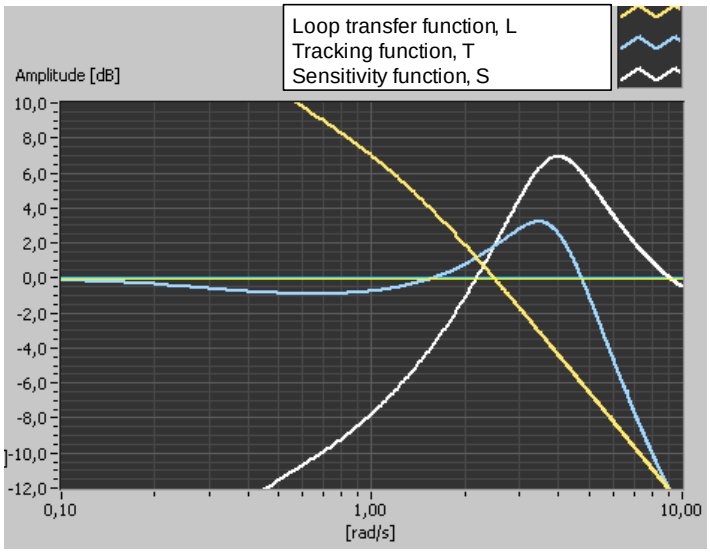

Figure 3.3:

4. Estimate the response time, T_r, of the response on the process output variable due to a step change of the setpoint.

Exercise 3.4

1. The diagram to the left of Figure 3.4 shows a non-controlled thermal process which is a liquid tank with throughput and heating. Assume that the amplitude gain of the frequency response of the transfer function from inlet temperature T_{in} to outlet temperature T is as shown in the Bode diagram in Figure 3.5.

   Assume that T_{in} contains a frequency component of amplitude $A_{T_{in}}$ of frequency 0.1 rad/s. Calculate the amplitude A_T of the corresponding steady-state response in T.

2. The diagram to the right of Figure 3.4 shows a temperature control system of the process. Assume that the amplitude gain of the frequency response of the transfer function from inlet temperature T_{in} to outlet temperature T of the control system is as shown in the Bode diagram in Figure 3.6.

   Assume that that T_{in} contains a frequency component of amplitude $A_{T_{in}}$ of frequency 0.1 rad/s. What is the amplitude A_T of the corresponding steady-state response in T? Compare the answer with problem 1 above. Is there any improvement by using control?

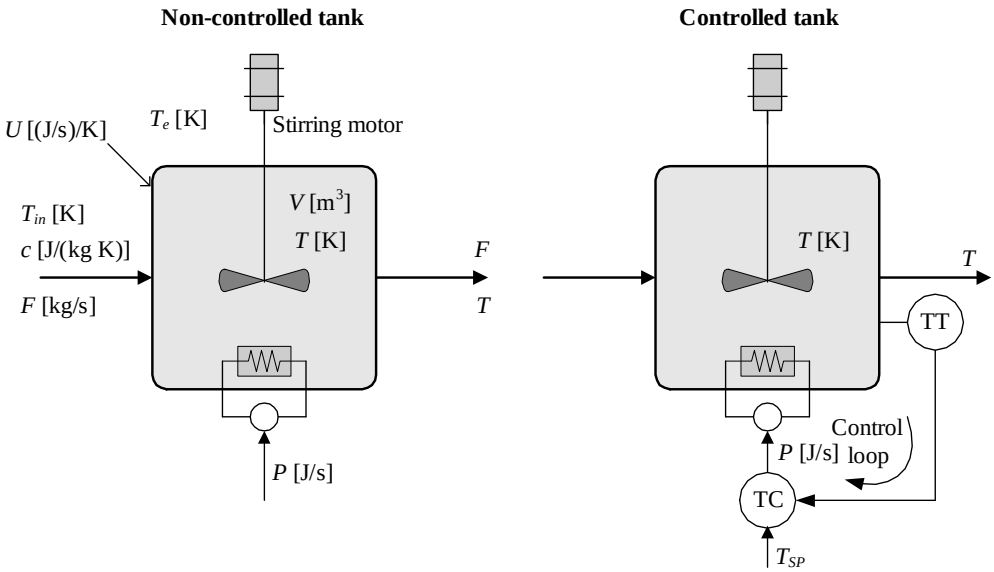

Figure 3.4:

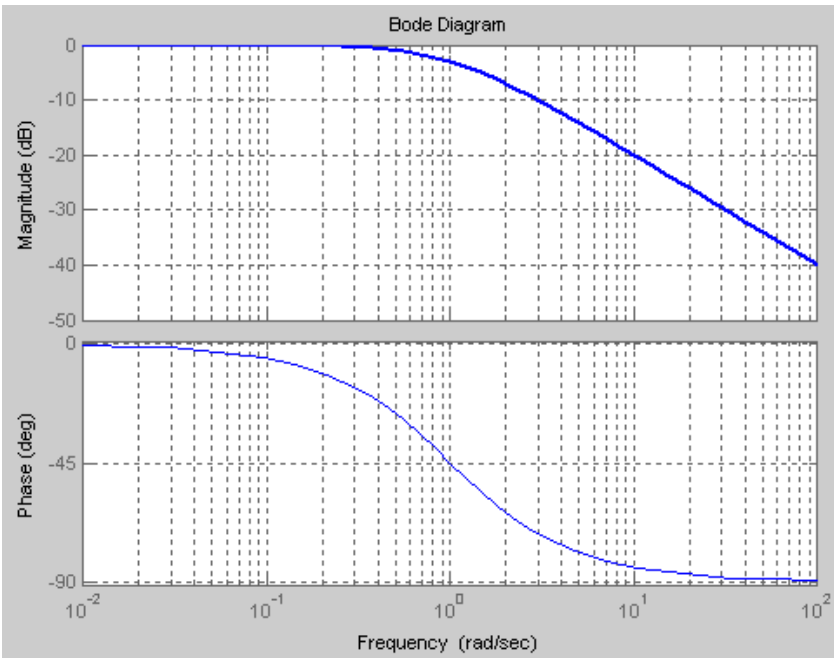

Figure 3.5:

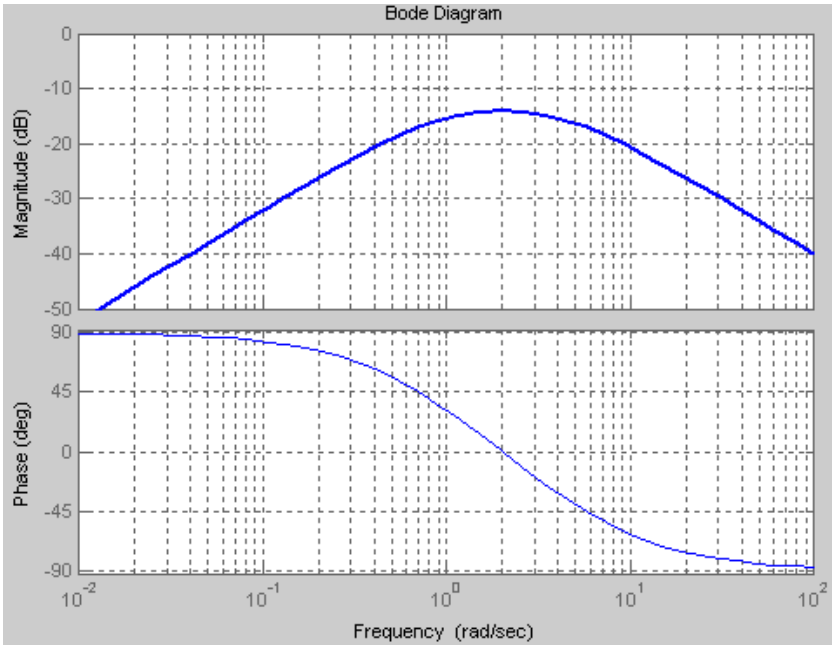
Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
-50
Phase (deg)
90
45
0
-45
-90
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Figure 3.6:

Chapter 4

Stability analysis of dynamic systems

4.1 Introduction

No exercises here.

4.2 Stability properties and impulse response

Exercise 4.1

Determine the stability property of the following transfer function by calculating its impulse response, $h(t)$.

$$H(s) = \frac{y(s)}{u(s)} = \frac{1}{s+1} \tag{4.1}$$

Also, make a rough sketch of $h(t)$.

To calculate $h(t)$, you can use the following Laplace transform:

$$\frac{k}{Ts+1} \iff \frac{ke^{-t/T}}{T} \tag{4.2}$$

4.3 Stability properties and poles

Exercise 4.2

Determine the stability property of the following transfer functions:

$$H_1(s) = \frac{1}{s+1} \tag{4.3}$$

$$H_2(s) = \frac{1-s}{1+s} \tag{4.4}$$

$$H_3(s) = \frac{1}{1-s} \tag{4.5}$$

$$H_4(s) = \frac{1}{(s+1)(s-1)} \tag{4.6}$$

$$H_5(s) = \frac{1}{s} \tag{4.7}$$

$$H_6(s) = \frac{1}{s^3} \tag{4.8}$$

$$H_7(s) = \frac{e^{-s}}{s+1} \tag{4.9}$$

$$H_8(s) = -\frac{1}{s+1} \tag{4.10}$$

$$H_9(s) = \frac{1}{s^2+s+1} \tag{4.11}$$

$$H_{10}(s) = \frac{1}{s^2+1} \tag{4.12}$$

$$H_{11}(s) = \frac{1}{(s+1)s} \tag{4.13}$$

4.4 Stability properties of state-space models

Exercise 4.3

Determine the stability property of the following state-space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{4.14}$$

Chapter 5

Stability analysis of feedback systems

5.1 Introduction

No exercises here.

5.2 Pole-based stability analysis of feedback systems

Exercise 5.1

Figure 5.1 shows a feedback control system. The transfer function of the

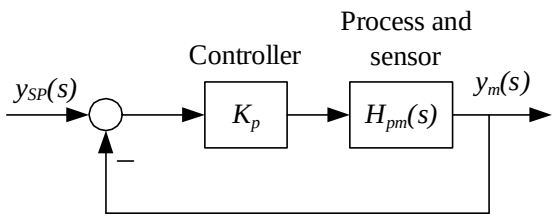

Figure 5.1:

process and sensor is

$$H_{pm}(s) = \frac{1}{s} \tag{5.1}$$

1. What is the stability property of $H_{pm}(s)$, i.e. of the process itself – also called the open-loop system?

2. For which values of the controller gain K_p is the control system asymptotically stable?

3. Has this exercise demonstrated that it is possible to obtain an asymptotically stable feedback system even though the process itself is asymptotically stable?

5.3 Nyquist's stability criterion

Exercise 5.2

Given a control system with loop transfer function

$$L(s) = \frac{K_p}{(s+1)^3 s} \tag{5.2}$$

Figure 5.2 shows the Nyquist curve of L with $K_p = 0.4$. (The curve

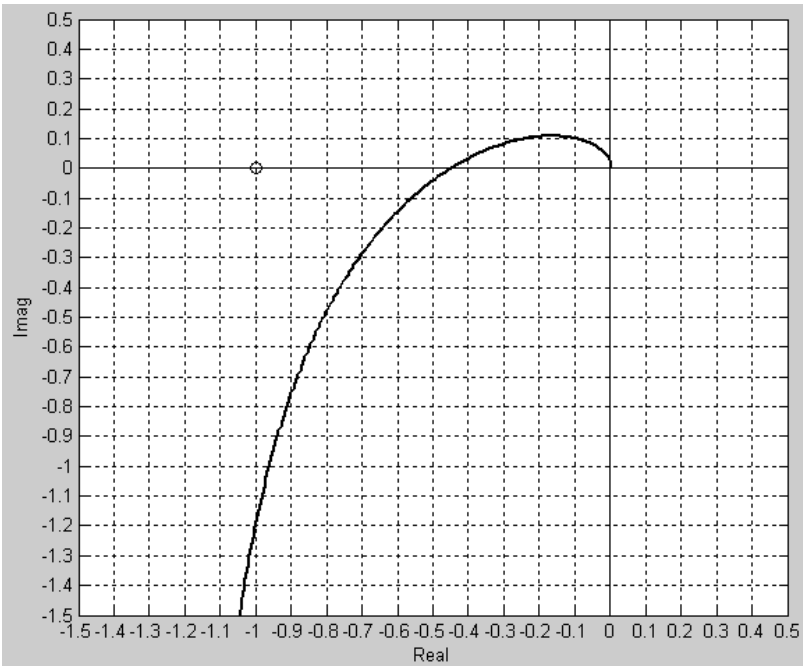

Figure 5.2:

actually encircles the whole right half plane.)

Use Nyquist's stability criterion to calculate the values of K_p that makes the control system become

- Asymptotically stable.
- Marginally stable.
- Unstable. In this case, what is the number of poles in the right half plane?

Exercise 5.3

Given a closed loop system having the following loop transfer function:

$$L(s) = \frac{K}{s-1} \tag{5.3}$$

1. Show that the corresponding open loop system is unstable by calculating the pole of the system.
2. Figure 5.3 shows the Nyquist curve of L with $K = 2$. Find using the

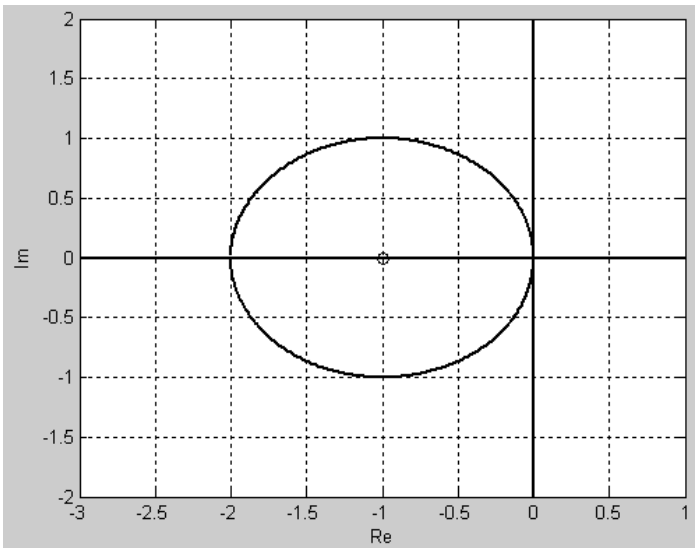

Figure 5.3:

Nyquist stability criterion for which values of K the closed loop

system is asymptotically stable. Confirm the answer by calculating the pole of the closed loop system.

Exercise 5.4

Figure 5.4 shows the Nyquist curve of $L(j\omega)$ of a feedback system which is open stable. The loop gain is then $K = 1$. For which values of K is the

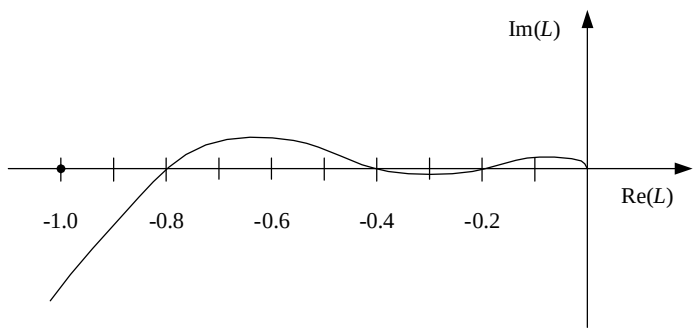

Figure 5.4:

feedback system asymptotically stable?

5.4 Stability margins

Exercise 5.5

Figure 5.5 shows the Nyquist curve of the loop transfer function L of an asymptotically stable control system. What is the gain margin GM and the maximum sensitivity gain $|S(j\omega)|_{\max}$?

Exercise 5.6

Figure 5.6 shows the amplitude gain curves of the loop transfer function L, the tracking function T and the sensitivity function S of a feedback control system. Determine the stability margin in terms of $|S(j\omega)|_{\max}$. Is the value in the range of reasonable values of this stability margin?

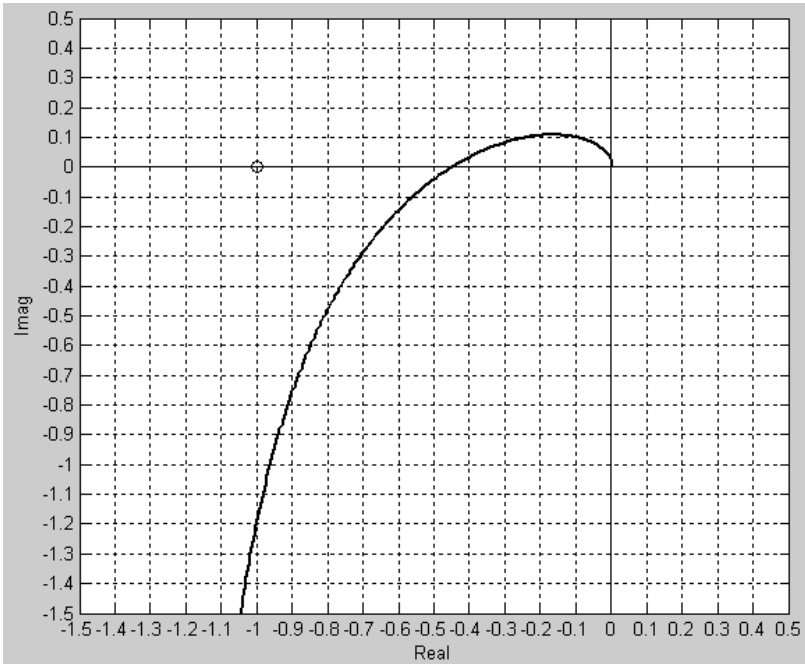

Figure 5.5:

5.5 Stability analysis in a Bode diagram

Exercise 5.7

Figure 5.7 shows the Bode diagram of the loop transfer function of a given control system.

1. Read off the stability margins GM and PM, and the crossover frequencies ω_c and ω_{180} in the Bode diagram.

2. How large increase of the loop gain will bring the system to the stability limit? What is the period T_p of the steady-state oscillations existing in the system when the system is at the stability limit.

5.6 Robustness in terms of stability margins

Exercise 5.8

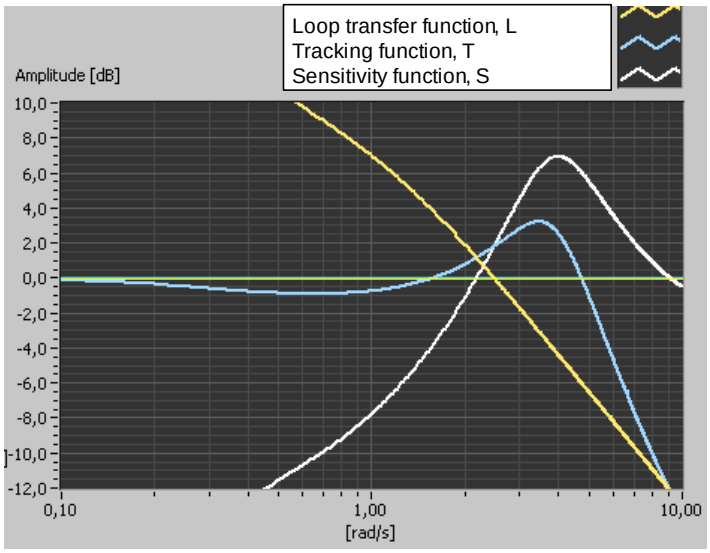

Figure 5.6:

Given a feedback control system with time delay $\tau = 4.2$ min. The control system has phase margin

$$PM = 45^\circ \tag{5.4}$$

and crossover frequency

$$\omega_c = 0.2 \text{ rad/min} \tag{5.5}$$

Assume that the time delay increases, but the controller parameters are not changed. With which value of the time delay τ is the control system marginally stable?

Exercise 5.9

Ziegler-Nichols' closed-loop method is based on bringing the closed loop to marginal stability with a P controller with a proper controller gain value. Explain in terms of frequency response why the Ziegler-Nichols' closed-loop method can not be used for tuning a PID controller for the following processes. It is assumed that the parameters of the transfer functions have positive values.

$$H_1(s) = \frac{K}{s} \text{ (integrator)} \tag{5.6}$$

$$H_2(s) = \frac{K}{Ts+1} \text{ (1. order system)} \tag{5.7}$$

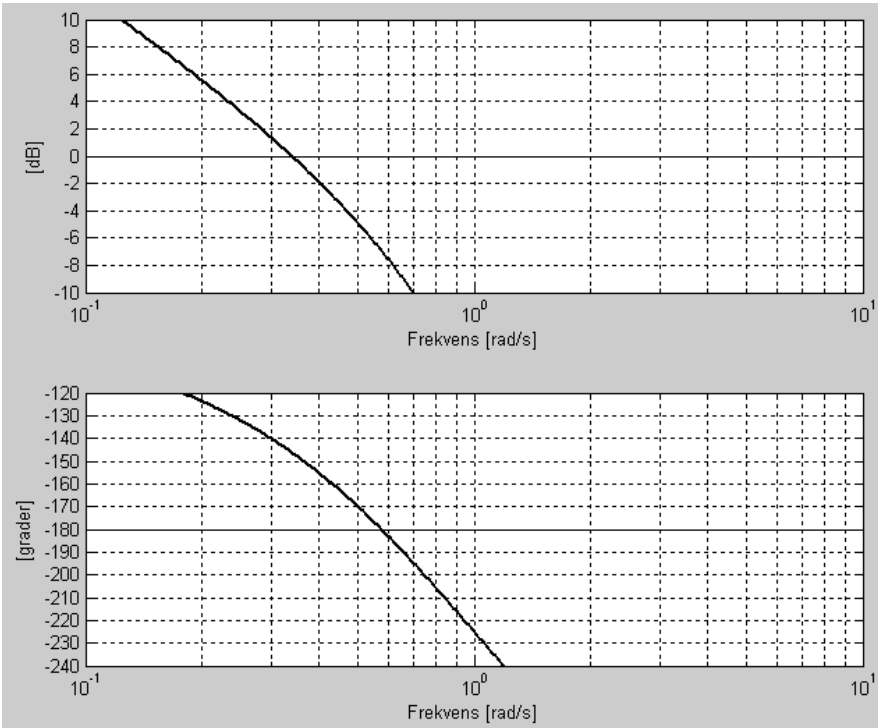

Figure 5.7:

$$H_3(s) = \frac{K{\omega_0}^2}{s^2 + 2\zeta\omega_0 + {\omega_0}^2} \text{ (2. order system)} \tag{5.8}$$

Chapter 6

Discrete-time signals

Exercise 6.1

Given the following continuous-time signal (a ramp):

$$x_c(t) = 2t \tag{6.1}$$

where t is time in seconds.

1. Assume that the signal is sampled with sampling time (time-step) $T_s = 0.5$ s. Express x_d as a function of the discrete time t_k. Write the discrete signal or sequence (time series) x_d from time 0 to 2. Plot x with both discrete time t_k and time index k along the abscissa.

2. Repeat Problem 1 above, but now with $T_s = 0.1$ s.

Chapter 7

Difference equations

7.1 Difference equation models

Exercise 7.1

Given the following difference equation:

$$y(k+3) + ay(k+1) = b_1 u(k+2) + b_0 u(k) \tag{7.1}$$

Write the corresponding difference equation having only zero or negative time shifts.

Exercise 7.2

Particularly in the area of discrete-time (digital) signal processing the difference equations constituting the mathematical model of signal filters are represented with mathematical *block diagrams*, in the same way as differential equations of continuous-time systems are represented with block diagrams.

Figure 7.1 shows the most frequently used blocks – or the *elementary blocks* – used in block diagrams of difference equation models.

A comment about the time delay block: The output $y(k)$ is equal to the time delayed input, $y(k-1)$:

$$y(k-1) = z^{-1} y(k) \tag{7.2}$$

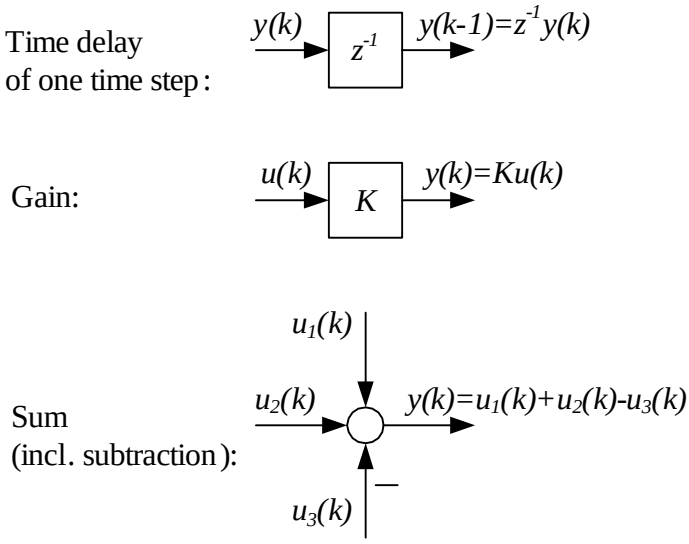

Figure 7.1: Elementary blocks for drawing block diagrams of difference equation models

Or, equivalently:

$$y(k) = z^{-1}y(k+1) \tag{7.3}$$

The operator z^{-1} is here a time-step delay operator, which is actually the z-transfer function of the time-step delay. (z-transfer functions are described in Chapter 11 of the text-book.)

Draw a block diagram of the following difference equation:

$$y(k+1) = ay(k) + bu(k) \tag{7.4}$$

where a and b are constant parameters. The block diagram shall have $u(k)$ as input and $y(k)$ as output.

7.2 Calculating responses from difference equation models

Exercise 7.3

Given the signal filter

$$y(k) = \frac{1}{3}\left[u(k) + u(k-1) + u(k-2)\right]$$

which is a moving average lowpass filter.

1. What is the steady-state response in y when the input u is a constant? Does the filter let a constant input pass unchanged, in steady-state?

2. Assume that the filter input u is a ramp:

$$\{u(k)\} = \{u(0),\ u(1),\ u(2),\ u(3),\ u(4)\} \tag{7.5}$$
$$= \{0,\ 0.5,\ 1.0,\ 1.5,\ 2.0\} \tag{7.6}$$
$$= \{0.5k\} \tag{7.7}$$

   Calculate the response y at $k = 0..4$. (You can assume that u is zero at negative k.)

   Also, calculate the general response $y(k)$. Will there be a constant difference from zero between the output and the input as time index goes to infinity?

Chapter 8

Discretizing continuous-time models

8.1 Simple discretization methods

Exercise 8.1

In Section 8.1 in the text-book the Forward discretization method and the Backward discretization method are based on *differentiation* approximations. However, they can be interpreted as *integration* approximations, too. The Tustin's method of discretization can also (easily) be described as an integration approximation method.

Given the following differential equation:

$$\dot{x} = f(x,\ u) \tag{8.1}$$

u is input variable. x is the state variable. f is some function – linear or nonlinear. The state variable at discrete time t_k is found by integration:

$$x(t_k) = x(t_{k-1}) + \int_{t_{k-1}}^{t_k} f\ d\tau \tag{8.2}$$

The integral in (8.2) can be approximated in several ways:

- Forward approximation:

$$x(t_k) \approx x(t_{k-1}) + T_s f\left[x(t_{k-1}), u(t_{k-1})\right] \tag{8.3}$$

- Backward approximation:

$$x(t_k) \approx x(t_{k-1}) + T_s f\left[x(t_k), u(t_k)\right] \tag{8.4}$$

- Tustin's approximation:

$$x(t_k) \approx x(t_{k-1}) + \frac{T_s}{2}\left\{f\left[x(t_k), u(t_k)\right] + f\left[x(t_{k-1}), u(t_{k-1})\right]\right\} \tag{8.5}$$

Illustrate each of the above approximations in a figure. You can base your drawing on the the sketch shown in Figure 8.1.

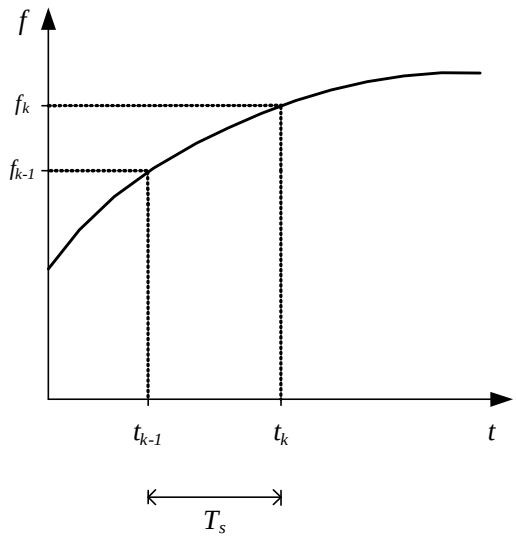

Figure 8.1:

8.2 Discretizing a simulator of a dynamic system

Exercise 8.2

See Exercise 1.1 which shows a mathematical model of two coupled liquid tanks. Develop a simulation algorithm for levels h_1 and h_2 based on Forward discretization. Why is is better to apply Forward discretization than Backward discretization in this example?

8.3 Discretizing a signal filter

Exercise 8.3

The transfer function of a first order highpass filter is

$$H(s) = \frac{y(s)}{u(s)} = \frac{T_f s}{T_f s + 1} \tag{8.6}$$

where T_f [s] is the filter constant. Derive a discrete-time filter algorithm in the form of a difference equation relating the output y and the input u based on Backward differentiation. The time-step is T_s [s].

8.4 Discretizing a PID controller

Exercise 8.4

In the text-book the continuous-time PID control function is discretized by discretizing the time-differentiated control function. An alternative way of obtaining a discrete-time PID control function is to discretize the P term, the I term, and the D term (in a parallel or additive PID controller) individually, and then summing the individual discrete-time terms. This is what this exercise is about.

For simplicity a PI (not a PID) controller will be discretized here.

The starting point is the continuous-time PI control function:

$$u = u_0 + \underbrace{K_p e}_{u_p} + \underbrace{\frac{K_p}{T_i} \int_0^t e \, d\tau}_{u_i} \tag{8.7}$$

1. Discretize (8.7) according to the alternative method describes above. The I (integral) term u_i can be discretized with the same method as used in the text-book: Discretize the time-differentiated version of the I term using the Backward method.
2. Suggest a way to implement integral anti windup in your discrete-time PI controller.

Exercise 8.5

As you will see in this exercise, you may get a surprise when a discrete-time P controller operates in steady state!

Assume that the controller is originally a standard discrete-time PID controller as developed in the text-book:

$$
\begin{aligned}
u(t_k) &= u(t_{k-1}) + [u_0(t_k) - u_0(t_{k-1})] && (8.8)\\
&\quad + K_p\,[e(t_k) - e(t_{k-1})] && (8.9)\\
&\quad + \frac{K_p T_s}{T_i} e(t_k) && (8.10)\\
&\quad + \frac{K_p T_d}{T_s}\,[e(t_k) - 2e(t_{k-1}) + e(t_{k-2})] && (8.11)
\end{aligned}
$$

1. What is the corresponding discrete-time P controller (including the manual control signal)? (Just remove the integral term and the derivative term by setting $T_i = \infty$ and $T_d = 0$, respectively).

2. Assume that the control system is in steady state (all signals being constant), and that the (steady state) control error for some reason is different from zero, say e_s. Show that the control error is *not* reduced even if the controller gain is increased (you would probably expect the the error to be reduced if the controller gain is increased).

3. Now, turn the controller into an I controller, with constant manual control signal. (The points of this task appear clearer with an I controller than with a PI or a PID controller.) Show that the control signal will change only as long as the control error is different from zero.

Exercise 8.6

Assume a process which is to be controlled with a discrete-time PID controller. The process has response time of 50 sec. What is the maximum time-step or sampling time that the controller should use? A typical samling time in industrial controllers is 0.1 s. Is this ok for the given process?

Exercise 8.7

Figure 8.2 shows an air heater. A fan with fixed speed blows air through the pipe. The air is heated by a electric heater. The control signal u is the

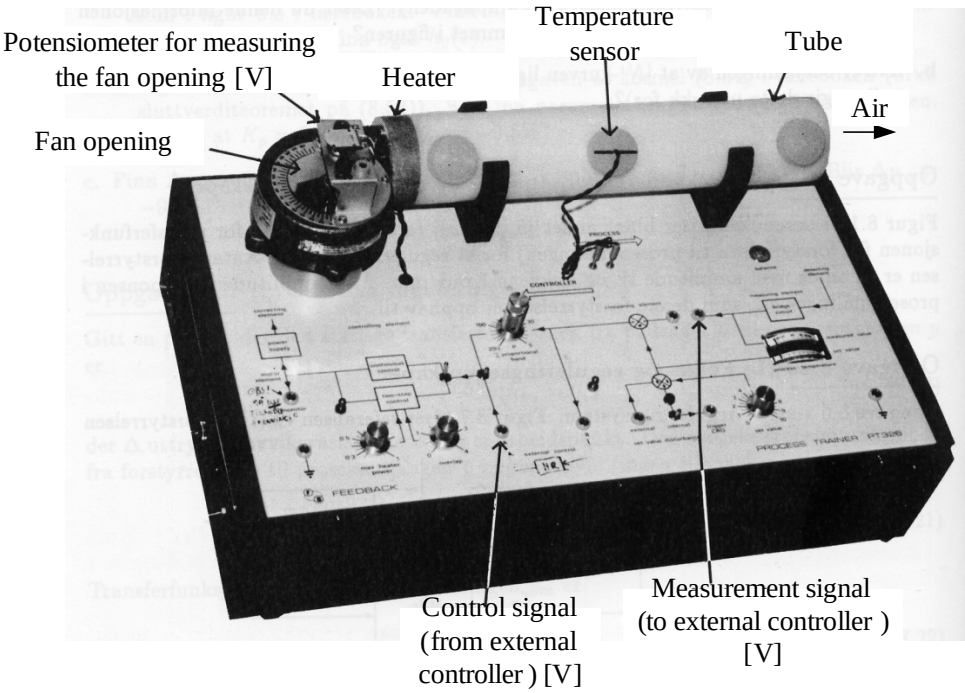

Figure 8.2:

voltage signal which controls (adjusts) the power supplied to the heater. The temperature is controlled with a discrete-time PI controller implemented in a PC.

Figure 8.3 shows the temperature reference and measurement and the control signal with different time-steps T_s.

1. What is the time-step T_s in case B and C in Figure 8.3?
2. Why does the control system get reduced stability as the time-step is increased?

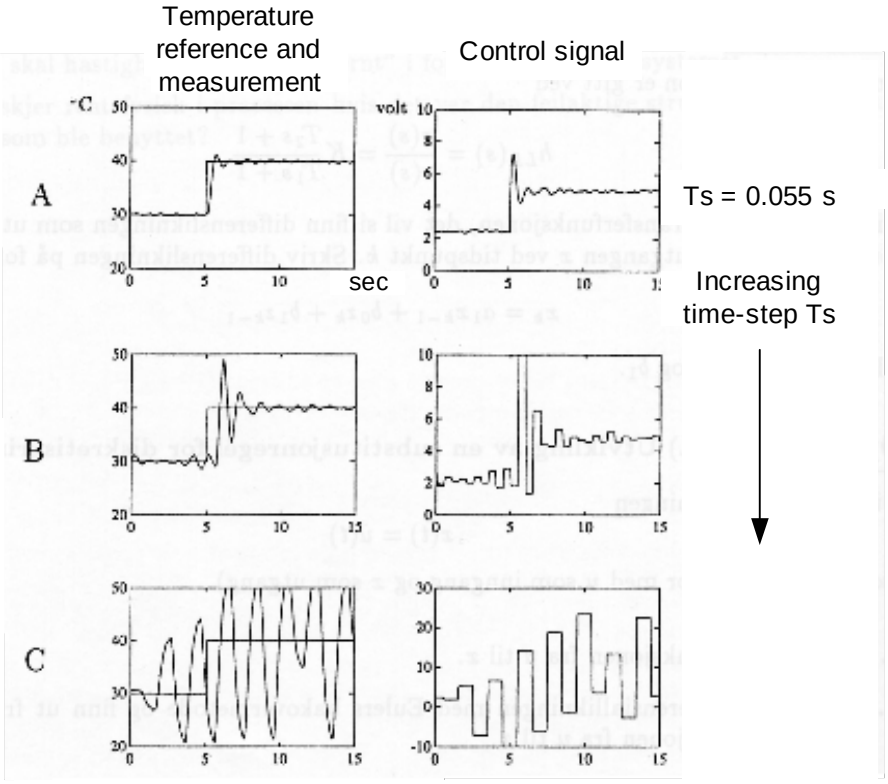
Temperature reference and measurement
Control signal
°C
volt
A
B
C
sec
Ts = 0.055 s
Increasing time-step Ts

Figure 8.3:

Chapter 9

Discrete-time state space models

9.1 General form of discrete-time state space models

Exercise 9.1

In Exercise 1.1 a mathematical model of two coupled liquid tanks is presented. The model written as a continuous-time state space model is

$$\dot{h}_1 = \frac{1}{A_1}\left(K_p u_1 - K_{v_1}\sqrt{\frac{\rho g h_1}{G}}\right) \tag{9.1}$$

$$\dot{h}_2 = \frac{1}{A_2}\left(K_{v_1}\sqrt{\frac{\rho g h_1}{G}} - K_{v_2} u_2 \sqrt{\frac{\rho g h_2}{G}}\right) \tag{9.2}$$

By applying Forward differentiation approximation to the time-derivatives, we get the following discrete-time model:

$$\dot{h}_1(t_k) \approx \frac{h_1(t_{k+1}) - h_1(t_k)}{T_s} = \frac{1}{A_1}\left(K_p u_1(t_k) - K_{v_1}\sqrt{\frac{\rho g h_1(t_k)}{G}}\right) \tag{9.3}$$

$$\dot{h}_2(t_k) \approx \frac{h_2(t_{k+1}) - h_2(t_k)}{T_s} = \frac{1}{A_2}\left(K_{v_1}\sqrt{\frac{\rho g h_1(t_k)}{G}} - K_{v_2} u_2(t_k)\sqrt{\frac{\rho g h_2(t_k)}{G}}\right) \tag{9.4}$$

Assume that both levels are output variables. Write this model as a discrete-time state space model on the standard form

$$h_1\left(t_{k+1}\right) = f_1\left[h_1(t_k), h_2(t_k), \cdots\right] \tag{9.5}$$

$$h_2\left(t_{k+1}\right) = f_2\left[h_1(t_k), h_2(t_k), \cdots\right] \tag{9.6}$$

(which corresponds to the compact standard form $x\left(t_{k+1}\right) = f\left[x(t_k), \cdots\right]$). Is the state space model linear or nonlinear?

9.2 Linear discrete-time state space models

Exercise 9.2

Write the following difference equations model as a state space model on matrix-vector form.

$$x_1(k+1) = -0.5x_1(k) \tag{9.7}$$

$$x_2(k+1) = 2u(k) - x_2(k) - 3x_1(k) \tag{9.8}$$

$$y(k) = x_2(k) + 4u(k) \tag{9.9}$$

9.3 Discretization of continuous-time state space models

Exercise 9.3

Consider the nonlinear continuous-time state space model (9.1) – (9.2). According to Exercise 9.1: If you discretize this model using Forward differentiation approximation, the resulting discrete-time model can be written on the following standard state-space model form:

$$h_1\left(t_{k+1}\right) = f_1\left[h_1(t_k), h_2(t_k), \cdots\right] \tag{9.10}$$

$$h_2\left(t_{k+1}\right) = f_2\left[h_1(t_k), h_2(t_k), \cdots\right] \tag{9.11}$$

What if you instead use the Backward differentiation approximation?

Exercise 9.4

1. Can you discretize the following (first order) state-space model using the ZOH (Zero Order Hold) method? (Yes or no.)

$$\dot{x} = -K_1 \left|x\right| x + K_2 u \tag{9.12}$$

   (K_1 and K_2 are constants.)

2. It can be shown that a linearizing (9.12) gives the following state-space model:

$$\Delta\dot{x} = -2K_1 \left|x\right| \Delta x + K_2 \Delta u \tag{9.13}$$

   Can this model be discretized using the ZOH method? (Yes or no.)

Chapter 10

The z-transform

10.1 Definition of the z-transform

Exercise 10.1

Calculate the $\mathcal{Z}$-transform of $\delta(k)$ which a unit impulse. This is a signal having amplitude 1 at discrete time $k = 0$ and zero at other points of time.

10.2 Properties of the z-transform

Exercise 10.2

The Linearity property of the $\mathcal{Z}$-transform can be expressed as

$$k_1 y_1(z) + k_2 y_2(z) \Longleftrightarrow k_1 y_1(k) + k_2 y_2(k) \tag{10.1}$$

Show that the Linearity property holds for this example of a signal:

$$y(k) = A + B = AS(k) + BS(k) \tag{10.2}$$

where A and B are constants, and $S(k)$ is the unity step functions, i.e. a step occuring at time zero. (Hint: Show that the $\mathcal{Z}$-transform of the right side of (10.1) is equal to the left side of (10.1) for the given signal.)

10.3 z-transform pairs

Exercise 10.3

Calculate the $\mathcal{Z}$-transform of a step signal of amplitude 4 which occurs at time-index 2. You can use $S(k)$ to represent a unit step function, i.e. a step occuring at time zero.

10.4 Inverse z-transform

Exercise 10.4

Calculate the inverse $\mathcal{Z}$-transform of

$$y(z) = \frac{z}{z - 0.5} \tag{10.3}$$

Chapter 11

Discrete-time (or z-) transfer functions

11.1 Introduction

No exercises here.

11.2 From difference equation to transfer function

Exercise 11.1

Given the signal filter

$$y(k) = \frac{1}{3}\left[u(k) + u(k-1) + u(k-2)\right]$$

which is a moving average lowpass filter. Find the z transfer function from u to y.

11.3 From transfer function to difference equation

Exercise 11.2

Assume that the following transfer function from input signal u to output signal y of a physical system is found by some system identification method:

$$H(z) = \frac{y(z)}{u(z)} = \frac{a}{bz^2 + cz + d}$$

What is the corresponding difference equation relating u and y?

11.4 Calculating time responses for discrete-time transfer functions

Exercise 11.3

Assume that the input signal u to the following transfer function is an impulse of amplitude A.

$$H(z) = \frac{y(z)}{u(z)} = \frac{z}{z-1} \tag{11.1}$$

Calculate the output response $y(k)$.

11.5 Static transfer function and static response

Exercise 11.4

In Exercise 11.1 the following transfer function of a moving avarage lowpass filter was found:

$$H(z) = \frac{y(z)}{u(z)} = \frac{1}{3}\left[1 + z^{-1} + z^{-2}\right] \tag{11.2}$$

1. Calculate the corresponding static transfer function H_s.
2. Assume that the filter input is a constant, $u(k) = U$. Calculate the corresponding steady-state filter output y_s from H_s.

11.6 Poles and zeros

Exercise 11.5

Given the following transfer function:

$$H(z) = \frac{bz^{-2} + z^{-1}}{1 - az^{-1}} \tag{11.3}$$

Calculate the poles and the zeros of the transfer function.

11.7 From s-transfer functions to z-transfer functions

Exercise 11.6

The s-transfer function of a (continuous-time) integrator is

$$H_{\text{cont}}(s) = \frac{y(s)}{u(s)} = \frac{1}{s} \tag{11.4}$$

Derive a corresponding z-transfer function $H_{\text{disc}}(z)$ assuming Backward discretization. The time-step is T_s.

Chapter 12

Frequency response of discrete-time systems

Exercise 12.1

Given the following transfer function:

$$H(z) = \frac{1}{z} \tag{12.1}$$

What is the amplitude gain function and the phase lag function?

Exercise 12.2

Given a continuous-time filter with transfer function

$$H_{cont}(s) = \frac{1}{T_f s + 1} \tag{12.2}$$

with

$$T_f = 1 \text{ sec} \tag{12.3}$$

Discretization of $H_{cont}(s)$ using the Backward method of discretization with time-step T_s gives the following discrete-time filter:

$$H_{disc}(z) = \frac{az}{z - (1 - a)} \tag{12.4}$$

where

$$a = \frac{T_s}{T_f + T_s} \tag{12.5}$$

Figure 12.1 shows the frequency responses of $H_{cont}(s)$ and $H_{disc}(z)$ with time-step

$$T_s = 0.2 \text{ sec} \tag{12.6}$$

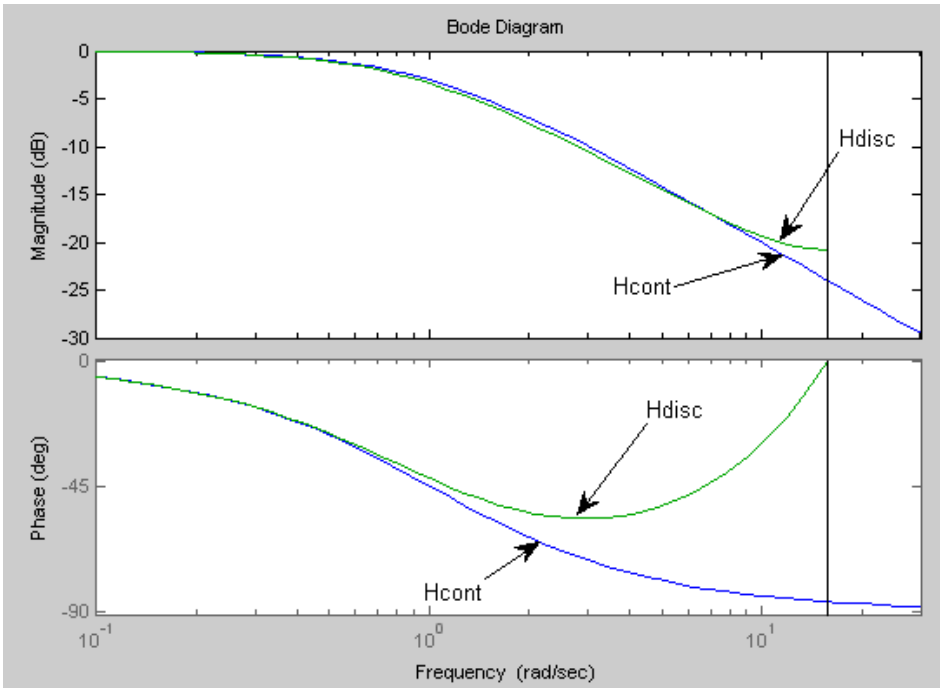

Figure 12.1:

1. Why is there a difference between the frequence responses of $H_{cont}(s)$ and $H_{disc}(z)$, and why is the difference more apparent at higher frequencies than at lower frequencies? (Qualitative answers are ok.)

2. The frequency response curves of $H_{disc}(z)$ are unique up to a certain frequency – which frequency? Express this frequency in Hz and rad/s. Is that frequency indicated in Figure 12.1?

Chapter 13

Stability analysis of discrete-time dynamic systems

13.1 Definition of stability properties

Exercise 13.1

Given the following transfer function:

$$H(z) = \frac{y(z)}{u(z)} = \frac{1}{z - 0.5} \tag{13.1}$$

Calculate the first four values of the impulse response. Determine the stability property from the impulse response.

13.2 Stability analysis of transfer function models

Exercise 13.2

Determine the stability property of the following transfer functions:

$$H_1(s) = \frac{1}{z - 0.5} \tag{13.2}$$

$$H_2(s) = \frac{1}{z + 0.5} \tag{13.3}$$

$$H_3(s) = \frac{z-2}{z-0.5} \tag{13.4}$$

$$H_4(s) = \frac{1}{z-1} \tag{13.5}$$

$$H_5(s) = \frac{1}{z-2} \tag{13.6}$$

$$H_6(s) = \frac{1}{(z-1)^2} \tag{13.7}$$

$$H_7(s) = \frac{1}{z} \tag{13.8}$$

$$H_8(z) = \frac{1}{z^2 - 2.5z + 1} \tag{13.9}$$

Exercise 13.3

Discretizing the continuous-time transfer function

$$H_{\text{con}}(s) = \frac{y(s)}{u(s)} = \frac{K}{Ts+1} \tag{13.10}$$

using the Forward discretization method with time-step T_s yields the following discrete-time transfer function:

$$H_{\text{dis}}(z) = \frac{y(z)}{u(z)} = \frac{\frac{KT_s}{T}}{z - \left(1 - \frac{T_s}{T}\right)} \tag{13.11}$$

For which (positive) values of T_s is the discrete-time system asymptotically stable? (You can assume that T is positive.)

13.3 Stability analysis of state space models

Exercise 13.4

Determine the stability property of the following state space model.

$$x(k+1) = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{13.12}$$

Chapter 14

Analysis of discrete-time feedback systems

Exercise 14.1

Given a feedback control system where the controller (a P controller)

$$H_c(z) = K_p \tag{14.1}$$

controls a process. The transfer function of the combined process and sensor (these systems are in series) is

$$H_{pm}(z) = \frac{z^{-1}}{1 - z^{-1}}$$

Calculate for which values of the controller gain K_p the control system is asymptotically stable.

Exercise 14.2

Assume that Figure 14.1 shows the Bode diagram of the loop transfer function of some discrete-time control system.

Read off the stability margins GM and PM, and the crossover frequencies ω_c and ω_{180} in the Bode diagram.

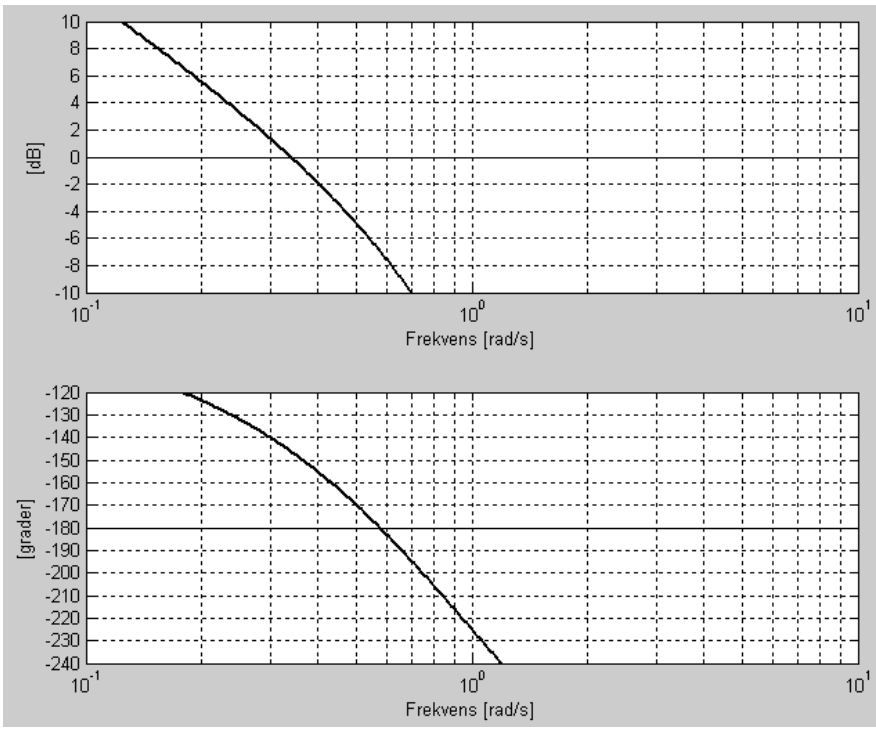
10
8
6
4
2
0
-2
-4
-6
-8
-10
[dB]
10^{-1}
10^{0}
10^{1}
Frekvens [rad/s]
-120
-130
-140
-150
-160
-170
-180
-190
-200
-210
-220
-230
-240
[grader]
10^{-1}
10^{0}
10^{1}
Frekvens [rad/s]

Figure 14.1:

Chapter 15

Stochastic signals

15.1 Introduction

No exercises here.

15.2 How to characterize stochastic signals

Exercise 15.1

Given the following sequence of measurement values:

$$\{x(k)\} = \{x(0), x(1), x(2)\} = \{0.73, 1.23, 0.89\} \tag{15.1}$$

1. Calculate the mean value, m_x.
2. Calculate the variance, σ_x^2.
3. Calculate the standard deviation, σ_x.
4. Calculate the auto-covariance, $R_x(L)$ with $L = 0$ and $L = 1$ for the following options (the same options that are defined for cross-covariance in the text-book):
   (a) Raw estimate
   (b) Normalized estimate
   (c) Unbiased estimate

(d) Biased estimate

Exercise 15.2

Given a random sequence, $\{x(k)\}$, uniformly distributed between $-A$ and $+A$.

1. Calculate the mean value or expectation value m_x using this formula:

$$m_x = \int_{-\infty}^{\infty} xP(x)dx = \int_{-A}^{A} xP(x)dx \tag{15.2}$$

   where $P(x)$ is the probability density. For a uniformly distributed signal,

$$P(x) = \frac{1}{2A} \tag{15.3}$$

   since the area under the $P(x)$-curve must be 1 (the width of the curve is 2A, so the height is $1/(2A)$ for the area to be 1).

2. Show that the variance of the sequence is

$$\sigma_x^2 = \frac{A^2}{3} \tag{15.4}$$

15.3 White and coloured noise

Exercise 15.3

Draw by hand the principal auto covariance of white noise with variance 4. What is the standard deviation σ_x of this signal?

Exercise 15.4

The following discrete-time first order lowpass filter is one example of a shaping filter:

$$x(k) = ax(k-1) + (1-a)v(k) \tag{15.5}$$

which is a discrete-time first order lowpass filter. x is the filter output, and $v(k)$ is the filter input, which is assumed to be white noise.

1. Explain (no calculations are needed) why $a = 0$ makes the output become white, and hence, there is no "shaping" through the filter.
2. Explain (no calculations are needed) why a between 0 and 1 makes the output become "coloured".

15.4 Propagation of mean value and co-variance through static systems

Exercise 15.5

Assume that you in a computer tool – e.g. LabVIEW or Matlab – are to generate a random signal y having mean value 3 and variance 4, and that the tool has a function that generates a random signal u of mean value 0 and variance 1. How can you obtain y from u? (Express y as a mathematical function of u.)

Chapter 16

Estimation of model parameters

16.1 Introduction

No exercises here.

16.2 Parameter estimation of static models with the Least squares (LS) method

Exercise 16.1

See Example 16.1 in the text-book. Draw the three data sets (points) in a cartesian diagram. Draw in the same diagram the straight line which is the LS solution. Also indicate the prediction errors with lines in the diagram.

Exercise 16.2

Assume that the parameters a and b in the differential equation

$$h(k) + a\sqrt{h(k-1)} = h(k-1) + bu(k-1) \tag{16.1}$$

is to be estimated with the Least Squares (LS) method. Assume that the following samples of the variables h and u exist:

$$\{h(0), h(1), h(2), h(3), h(4)\} \tag{16.2}$$

$$\{u(0), u(1), u(2), u(3), u(4)\} \tag{16.3}$$

Write the total regression model

$$Y = \Phi\theta \tag{16.4}$$

which makes the basis for the LS estimation. However, you shall *not* calculate the estimate in this Problem. The regression model contains only the samples of h and u that are available. (You are to find the vector Y, matrix Φ, and the vector θ.)

Exercise 16.3

This exercise demonstrates that a LS-estimate converges towards an erroneous value – in other words the estimate is biased – if the noise or the model error has a mean value different from zero.[1]

The constant K in the model

$$y = K \tag{16.5}$$

is be estimated. Assume that the real (true) value of K is K_0. The observation y_k is

$$y_k = K_0 + e_k \tag{16.6}$$

where e_k is noise (or equation error or model error or prediction error). Calculate the LS estimate of K (you can assume that there are N observations). Assume that y_k in (16.5) is given by (16.6).

Exercise 16.4

Figure 16.1 shows the criterion function V as a function of model order n in a fictitous problem about parameter estimation. Which order should be selected?

Exercise 16.5

This exercise is based on Example 16.1 in the text-book. To stick to standard nomenclature, let us here use y in stead of z which was used in the example.

[1] In general, the estimate will be biased if the noise is *coloured* (non-white).

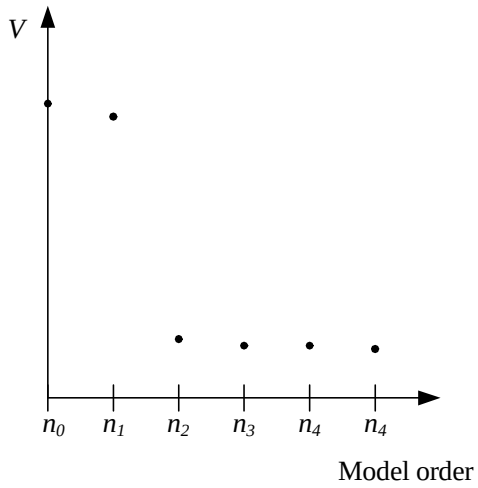

Figure 16.1:

In the example, parameters c and d in the model

$$y = cx + d \tag{16.7}$$

are fitted to the three data points

$$\begin{aligned}
(y_1, x_1) &= (0.8, 1.0) && (16.8)\\
(y_2, x_2) &= (3.0, 2.0) && (16.9)\\
(y_3, x_3) &= (4.0, 3.0) && (16.10)
\end{aligned}$$

The LS-solution is

$$\begin{bmatrix} c_{\mathrm{LS}} \\ d_{\mathrm{LS}} \end{bmatrix} = \begin{bmatrix} 1.6 \\ -0.6 \end{bmatrix} \tag{16.11}$$

1. Calculate the LS criterion

$$\begin{aligned}
V_{\mathrm{norm}} &= \frac{1}{m} V(\theta_{\mathrm{LS}}) && (16.12)\\
&= \frac{1}{m} \sum_{k=1}^{N} e_k^2 && (16.13)\\
&= \frac{1}{m} \sum_{k=1}^{N} (y_k - \varphi_k \theta_{LS})^2 && (16.14)\\
&= \frac{1}{m} \sum_{k=1}^{N} [y_k - (c_{LS} x_k + d_{LS})]^2 && (16.15)
\end{aligned}$$

for the data given above.

2. To indicate that (16.11) actually is the LS solution, you are now to calculate the criterion V_{norm} with some other values of c and d than the LS values. The question is: Will V_{norm} be larger with these non-LS values? You can (as an example) use the values

$$c = 2.2 \tag{16.16}$$

$$d = -1.4 \tag{16.17}$$

which gives a perfect match to the frist two of the three data sets:

$$(y_1, x_1) = (0.8, 1.0) \tag{16.18}$$
$$(y_2, x_2) = (3.0, 2.0) \tag{16.19}$$

Show that $c = 2.2$ and $d = -1.4$ is a perfect match to these two data sets.

Calculate V_{norm} given by (16.15) with $c = 2.2$ and $d = -1.4$. Has V_{norm} larger value than the LS-value of V_{norm} that you calculated in Problem 1 above? If yes, you have demonstrated that $c = 2.2$ and $d = -1.4$ are not LS parameters!

16.3 Parameter estimation of dynamic models

Exercise 16.6

Assume that you know in advance that a given physical process has time-delay of 2 sec. You are to estimate a discrete-time transfer function to from experimental input and output (measurement) sequences. The sampling time is 0.5 sec. You are not sure about the order of the transfer function, but what is the minimum order that it should have?

Exercise 16.7

Here is a mathematical model of an electric motor (a DC motor) with load torque:

$$J\dot{\omega} = \frac{K_T}{R_a}(v_a - K_e\omega) + T_L \tag{16.20}$$

Assume that the inertia J and the torque T_L will be estimated with the LS method. The motor is excited with the armature voltage v_a. The rotational speed ω is measured with a tachometer. K_T, R_a and K_e have known values.

Write the model on the standard form $y = \varphi\theta$. Use the center difference method to calculate $\dot{\omega}$.

Exercise 16.8

In e.g. subspace methods of system identification, the model that is estimated is always a linear model. If the model is actually nonlinear, a linear model will give a good representation of the system only near an operating point. How can you process the input and output data (the sequences) before using them in system identification to improve the accuracy of the estimated linear model, assuming the process is nonlinear?

Chapter 17

State estimation with observers

Exercise 17.1

Figure 17.1 shows an electric motor (a DC motor). It is manipulated with an input voltage signal, u, and the rotational speed is measured with a tachometer which produces a output voltage signal which is proportional to the speed. Assume that the speed, S [krpm = kilo revolutions per minute] is calculated continuously from the tachometer voltage. A proper

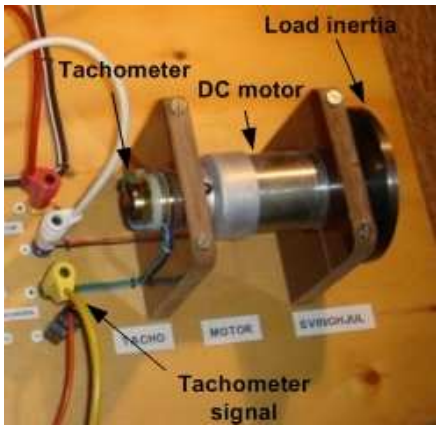

Figure 17.1:

mathematical model of the motor is

$$T_m \dot{S}(t) + S(t) = K_m[u(t) + L(t)] \tag{17.1}$$

L is the equivalent load torque (represented in the same unit as the control variable, namely voltage). L can be regarded as a process disturbance. K_m is gain. T_m is time-constant. (Model parameter values are given in Appendix A.1, but these values are not needed in the present exercise.)

The tasks below are about designing an observer which estimates the load L (and the speed S) based on the measurement of S. Assume that the value of L and how it varies is completely unknown, so the best assumption is to model it as an unknown constant.

1. Find the observer formulas for continuous-time implementation. (You can use S and L as names of the state variables.) Assume that the observer gain is given. Task 3 is about calculating this gain.
2. Find the observer formulas (algorithm) discrete-time implementation. The time-step is T_s. (Assume that the observer gain is given.)
3. Find the observer gains K_1 and K_2 as a function of the observer response-time T_r. (Use manual calculations.)
4. Check if the system is observable.
5. If you want to speed up the response of the estimates, would you then increase or descrease the observer response time T_r? Does this make the estimates more noisy or less noisy?
6. Assume that the estimates are too noisy due to measurement noise. How can you obtain smoother estimates without adjusting the parameters of the observer itself?
7. Assume that the load estimate, L_e, will be used in feedforward control of the motor speed. Derive the feedforward control function, i.e. $u_f = ...$
8. Assume that you want to build a speed control system with increased robustness against speed sensor failure. Explain how you can use the observer to obtain this.

Exercise 17.2

See Figure 17.1 in the text-book.

1. Explain that the estimation loop is based on updating the time-derivative of the estimates with a P (proportional) compensator or "controller".

2. Modeling errors, i.e. differences between the model assumed to represent the real system and the model used in the observer to calculate the state estimates, may cause non-zero steady-state errors in the state estimates with the P compensator. Think about basic controller theory. Suggest an alternative compensator that will cause the steady-state estimation error to become zero.

Exercise 17.3

This exercise is about using an observer for estimating acceleration from position measurement. In other words: You will design a *soft accelerometer*.

1. Write a state-space model with position, velocity, and acceleration as state variables. You can use the state names p, v, and a. Assume that you do not know how the acceleration varies, so the best assumption is that it's time-derivative is zero.

2. Derive the observer formulas for continuous-time implementation. Assume that the observer gain is given. Task 3 is about calculating this gain.

3. Derive the observer formulas (algorithm) discrete-time implementation. The time-step is T_s. (Assume that the observer gain is given.)

4. Find the observer gains K_1 and K_2 as a function of the observer response-time T_r. (Use manual calculations.)

Chapter 18

State estimation with Kalman Filter

Exercise 18.1

The system in this exercise is the same as in Exercise 17.1 about observer, namely an electric motor. Figure 18.1 shows the motor. It is manipulated with an input voltage signal, u, and the rotational speed is measured with a tachometer which produces a output voltage signal which is proportional to the speed. Assume that the speed, S [krpm = kilo revolutions per minute] is calculated continuously from the tachometer voltage. A proper

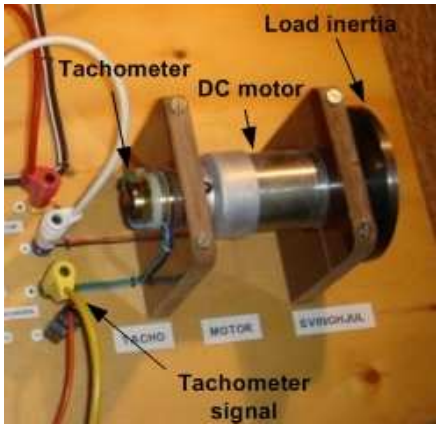

Figure 18.1:

mathematical model of the motor is

$$T_m \dot{S}(t) + S(t) = K_m[u(t) + L(t)] \tag{18.1}$$

L is equivalent load torque (represented in the same unit as the control variable, namely voltage). L can be regarded as a process disturbance. K_m is gain. T_m is time-constant. (Model parameter values are given in Appendix A.1, but these values are not needed in the present exercise.)

The tasks below are about designing a Kalman Filter which estimates the load L (and the speed S) based on the measurement of S. Assume that the value of L and how it varies is completely unknown, so the best assumption is to model it as an unknown constant.

1. Find the Kalman Filter formulas for estimating S and L. (You can use S and L as names of the state variables.) You can assume that the Kalman Filter gain K is given. Task 2 is about calculating this gain. The time step is T_s.
2. Define the quantities needed for calculating the steady-state Kalman Filter gain, K. (You are not required to calculate K. In practice you will use a proper function in e.g. Matlab or LabVIEW.)
3. Check if the system is observable.
4. How can you in a practical experiment find a reasonable value of the measurement noise covariance R?
5. How can you tune the Kalman Filter to speed up the response of the estimate of $L_{est} = L_c$? Is there any drawback related to increasing the response of the estimate?
6. Assume that the estimates are too noisy due to measurement noise. How can you obtain smoother estimates without adjusting parameters of the Kalman Filter itself?
7. Assume that the load estimate, $L_{est} = L_c$, will be used in feedforward control of the motor speed. Derive the feedforward control function, i.e. $u_f(t_k) = ...$
8. Assume that you want to build a speed control system with increased robustness against speed sensor failure. Explain how you can use the Kalman Filter to obtain this.

Chapter 19

Testing robustness of model-based control systems with simulators

Exercise 19.1

Figure 19.1 shows a model-based speed control system of an electric motor.

The motor is controlled with an input voltage signal, u, and the rotational speed, S, is measured with a tachometer which produces a voltage being proportional to the speed.

A proper mathematical model of the motor is the following differential equation:

$$\dot{S}(t) = \frac{1}{T_m}\left\{-S(t) + K_m[u(t) + L(t)]\right\} \tag{19.1}$$

L is equivalent load torque (represented in the same unit as the control variable, namely voltage). L can be regarded as a process disturbance. K_m is gain. T_m is time-constant. (Parameter values are given in Appendix A.1, but these values are not needed in the present exercise.)

The control system is based on feedback control and feedforward control. The feedback controller is a PI controller which is tuned using the Skogestad's model-based tuning formulas for a "time-constant process":

$$K_p = \frac{T_m}{K_m T_C} \tag{19.2}$$

$$T_i = \min\left[T_m,\, cT_C\right] \tag{19.3}$$

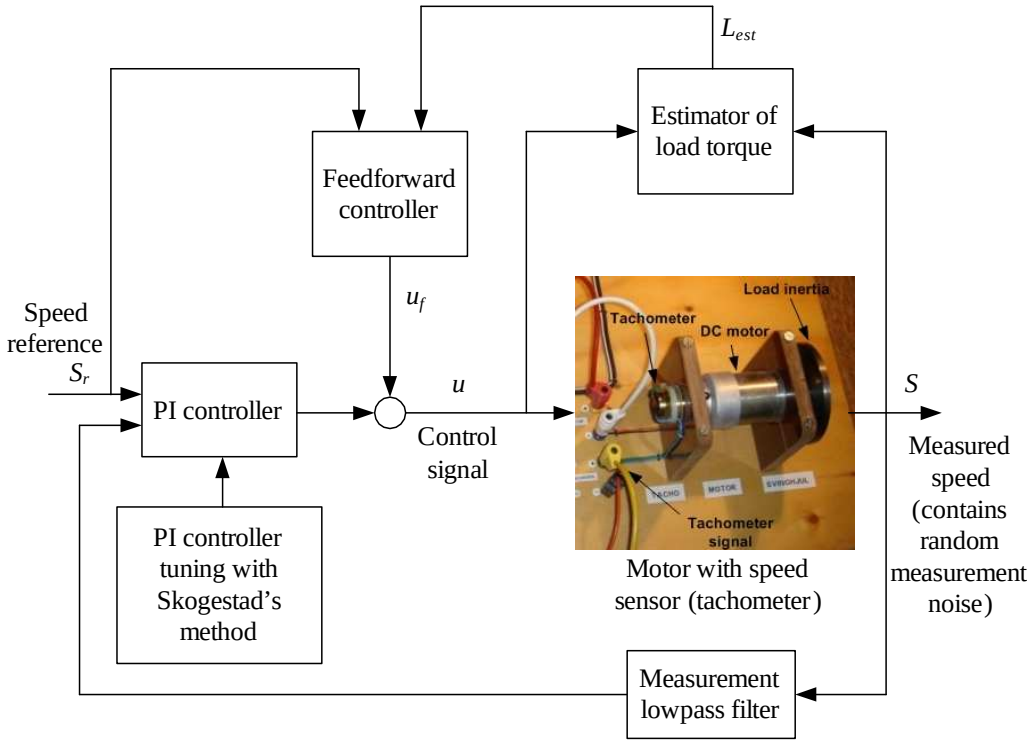

Figure 19.1:

where T_C is the specified closed-loop time-constant. We set $c = 2$. Assume (for simplicity) that it decided to use

$$T_i = T_m \tag{19.4}$$

The feedforward controller is based on the motor model (19.1) from estimated load torque L_{est} and speed reference S_r:

$$u_f(t) = \frac{1}{K_m} T_m \dot{S}_r(t) + S_r(t) - L_{est}(t) \tag{19.5}$$

L_{est} is estimated with an estimator (an observer or a Kalman Filter – which one of these does not matter here), which uses the mathematical model of the motor to calculate the estimate.

The feedback controller, the feedforward controller, and the estimator are model-based. Hence, the whole control system is *model-based*.

Assume that you intend to test the robustness of the model-based control system against model errors, and also that you want to see how the

measurement noise is influencing the behaviour of the control system. Unfortunately, you can not perform experiments on the real motor. In stead you must use a simulated motor. Explain how you can do this simulated experiments. Draw a block diagram similar to the one shown in Figure 19.1, where you indicate which model parameters to use in the individual blocks. You can assume that it is interesting to see if the control system is robust against model parameter variations of ± 20 %.

Chapter 20

Feedback linearization

Exercise 20.1

Figure 20.1 shows a tank where continuous flows of cold liquid and hot liquid are mixed in a tank.[1] The liquid in the tank is assumed being homogeneous. The product flow rate out of the tank is controlled with a ordinary flow control loop.

The level and the temperature of the tank shall be controlled to follow (track) their reference values. The cold liquid and the hot liquid flows can be manipulated, so they are control variables. (To make the actual flows become equal to the demanded flow as calculated by the multivariable controller, local flow control loops around the pumps may be needed, but these control loops are not shown in the figure.)

It is not obvious how to control the cold flow and the hot flow to obtain the reference level and temperature because both flows affect both the level and the temperature. The control problem may be solved with a "traditional" control structure where e.g. the level controller adjusts the cold flow, and the temperature controller adjusts the ratio between hot and cold flow. But, instead of such a traditional control structure, you will design a model-based multivariable controller based on feedback linearization.

The mathematical model of the process is as follows. The model is based on the following assumptions:

[1] This example is not very realistic, but it is assumed to be relatively simple and easy to understand.

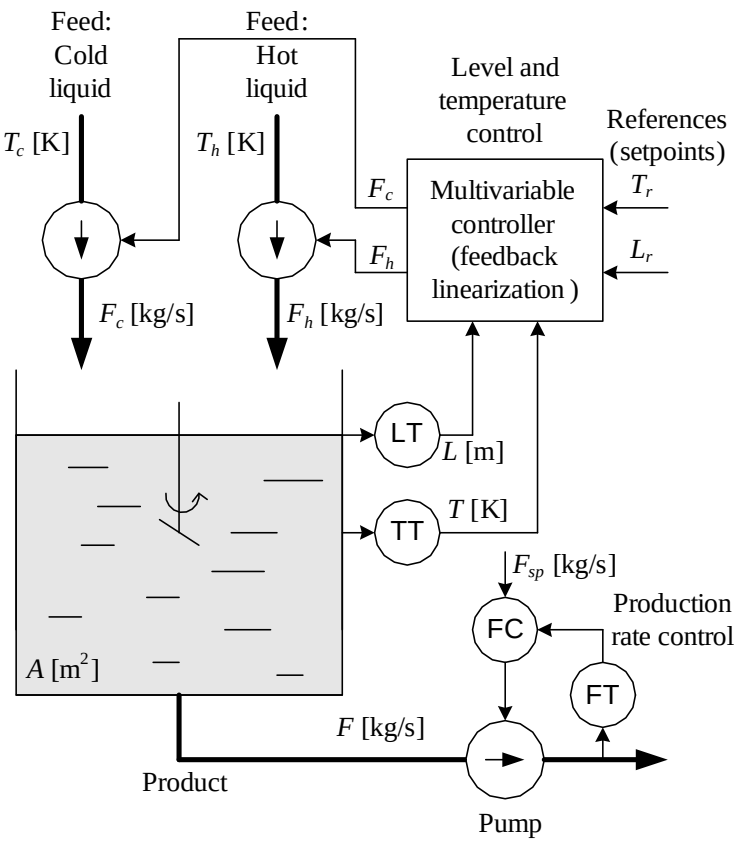

Figure 20.1:

- The density ρ and the specific heat capacity c are the same in all flows and in the tank.
- The temperature is homogeneuous in the liquid in the tank.
- There is no heat transfer through the walls of the tank.
- Energy dependent on pressure and kinetics is disregarded.

Mass balance of the mixed liquid of the tank is

$$\rho A\dot{h} = F_c + F_h - F \tag{20.1}$$

Energy balance of the liquid in the tank is (it is assumed that both the level and the temperature can vary)

$$c\rho A(\dot{LT}) = cF_cT_c + cF_hT_h - cFT \tag{20.2}$$

In (20.2),

$$c\rho A(\dot{LT}) = c\rho A\dot{L}T + c\rho AL\dot{T} \tag{20.3}$$

With (20.3) inserted into (20.2) and then cancelling c, (20.2) becomes

$$\rho AL\dot{T} = F_c\,(T_c - T) + F_h\,(T_h - T) \tag{20.4}$$

The process model is now (20.1) and (20.4).

1. Write the process model (20.1) and (20.4) on the standard form to be used in a feedback linearization controller where F_c and F_h are control variables, and L and T are state variables to be controlled (to track their respective references).

2. Design the control function based on feedback linearization, but you do not have to give the formulas for tuning the gains and the integral times of the internal PI controllers, as this is the topic of the following task. The references are L_r and T_r.

3. Calculate the gains, K_{p_L} and K_{p_L}, and the integral times, T_{i_L} and T_{i_T}, of the internal PI controllers. It is specified that the control loops of the decoupled processes (which are just integrators) shall have response times (time constants) T_{C_L} and T_{C_T}, respectively.

4. Which parameters and variables must be known at any instant of time to make the control function implementable?

5. Assume that there is a change in the level reference. Will this change cause any change in the temperature?

   Will a change in the temperature reference cause any change in the level?

6. Assume for example that any change in the temperature of the cold inflow, T_c, is regarded as a disturbance to the control system. Does the control function implement *feedforward* from this disturbance?

Exercise 20.2

Figure 20.2 shows a ship. In this exercise we concentrate on the so-called surge (forward-backward) direction, i.e., the movements in the other directions are disregarded. The wind acts on the ship with the force F_w

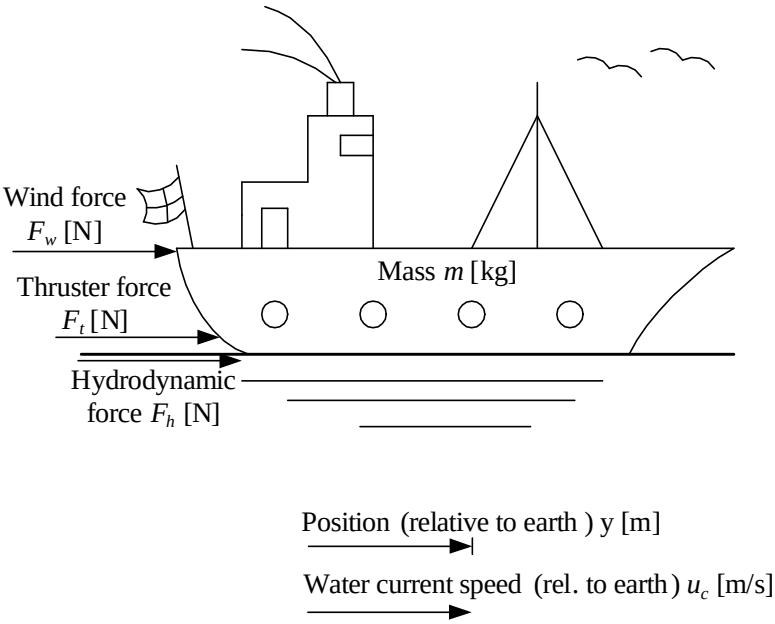

Figure 20.2:

which is a function of the wind attack angle ϕ and the wind speed V_w. This function is assumed to be known for a given ship. The hydrodynamic damping force F_h (damping from the water) is proportional to the square of the difference between the ship speed, $\dot{y}$, and the water current speed u_c. The proportionality constant is D.

Applying Newtons's Law of Motion we obtain the following mathematical model of the surge motion:

$$m\ddot{y} = \underbrace{-D|\dot{y} - u_c|\,(\dot{y} - u_c)}_{F_h} + F_w(\phi, V_w) + F_t \tag{20.5}$$

(Model parameter values are given in Appendix A.2, but these values are not needed in the present exercise.)

1. Design the position control function using feedback linearization. The position reference is r [m]. The closed loop time-constant is specified as T_C. (According to the comments about Skogestad's method for tuning a PID controller for a double-integrator, you can use $T_C/2$ in the Skogestad's formulas.) Express the PID parameters as a function of T_C.

2. Which variables and parameters must have known values (from

measurements or estimators) to make the controller function implementable.

3. Let's define the wind force F_w as a disturbance. Explain how the feedback linearization controller implements feedforward from this disturbance. Assuming that the ship model is correct and that F_w is perfectly known, what is then the impact that F_w will have on the ship position y?

Chapter 21

LQ (Linear Quadratic) optimal control

Exercise 21.1

See Figure 20.2 which shows a ship. In this exercise we concentrate on the so-called surge (forward-backward) direction, i.e., the movements in the other directions are disregarded. The wind acts on the ship with the force F_w which is a function of the wind attack angle ϕ and the wind speed V_w. This function is assumed to be known for a given ship. The hydrodynamic damping force F_h (damping from the water) is proportional to the square of the difference between the ship speed, $\dot{y}$, and the water current speed u_c. The proportionality constant is D.

Applying Newtons's Law of Motion we obtain the following mathematical model of the surge motion:

$$m\ddot{y} = \underbrace{-D|\dot{y} - u_c|\,(\dot{y} - u_c)}_{F_h} + F_w(\phi, V_w) + F_t \tag{21.1}$$

(Model parameter values are given in Appendix A.2, but these values are not needed in the present exercise.)

This exercise is about ship position control using LQ optimal control. Assume that the water current u_c has a known value at any instant of time (in a practical application it may have been estimated with a state estimator, e.g. a Kalman Filter). Also, assume that the wind force F_w has a known value at any instant of time (with a mathematical wind model the wind force can be calculate from information about the wind attack angle

ϕ and the wind speed V_w provided by the sensor). We also assume that the ship position y and speed $\dot{y}$ are known at any instant of time.

1. Write the ship model as a (nonlinear) state-space model using $x_1 = y$ and $x_2 = \dot{y}$ as state variables. F_t is control variable.

2. The ship position will be controlled with LQ control with integral action. Figure 21.1 shows a block diagram of the ship. Enhance this

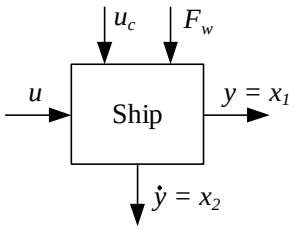

Figure 21.1:

block diagram so that it shows the control system in detail, including the feedbacks and the integrator of the controller.[1] You can assume that the controller gains have known values (calculation of these gains is the focus of a following task).

3. Augment the state-space model found in Problem 1 above with the state variable of the integral part of the controller. (This augmented state-space model is needed in the following problem.)

4. Figure 21.2 illustrates the matrices needed to compute the steady-state LQ controller gain G_s (using a proper function in e.g. Matlab or LabVIEW).Matrices A and B are found by linearizing the augmented state-space model found in the problem above. The elements of the weight matrices Q and R – at least their initial values since these values may be adjusted later – shall be expressed as functions of the allowable maximum values of the proper variable in the model.

5. Suppose you want to reduce the fuel consumption (less aggressive control). How can you adjust some of the weights of the LQ criterion to obtain this?

6. Suppose you want to increase the damping in the control system. In other words: You want to limit (reduce) the speed of the ship. How can you adjust some of the weights of the LQ criterion to obtain this?

[1] Feedforward from wind force F_w and water current u_c is an enhancement of the control system, but we will not include feedforward in this exercise.

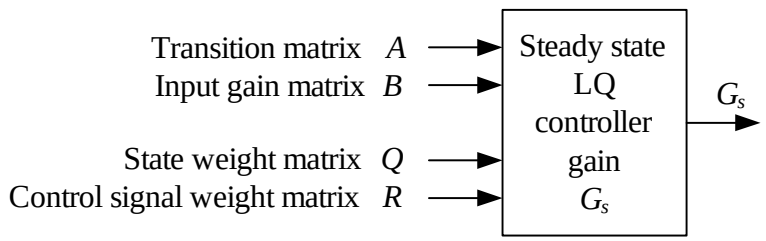

Figure 21.2: Information needed to compute the steady-state LQ controller gain, G_s.

Chapter 22

Model-based predictive control (MPC)

Exercise 22.1

If the setpoint profile is known in advance, the MPC controller will start adjusting the control signal before the setpoint is actually changed. (This is demonstrated in the text-book.) Assume that a PID controller with ordinary feedforward from setpoint is used in stead of MPC. Will then the control signal start increasing before the reference change?

Exercise 22.2

See Exercise 21.1 which is about positional control of a ship. In that exercise the position is controlled with LQ control. Now, assume MPC in stead of LQ control. The state variables are $x_1 = y$ and $x_2 = \dot{y}$. The control variable is F_t.

1. Assume that you want to make the control system more sluggish so that the positional control error is allowed to become larger. (One possible motivation may be that tight position control is not required for a period, for example if the ship is parked, waiting for a certain operation to be initiated.) Which parameter of the criterion is proper for adjustment, and should you increase of decrease that parameter?
2. The thruster force F_t has of course a positive limit and a negative limit (assuming that the thruster can rotate both directions). Can

the MPC algorithm take these limits into account when it calculates the optimal thruster force to be applied to the ship?

Exercise 22.3

MPC may be used for direct manipulation of the control variables. MPC can also be used in a hierarchic control system. Figure 22.1 shows using MPC in outer control loop while PID controllers are used in inner control loops. This control structure is similar to a conventional control structure frequently used in industry – which?

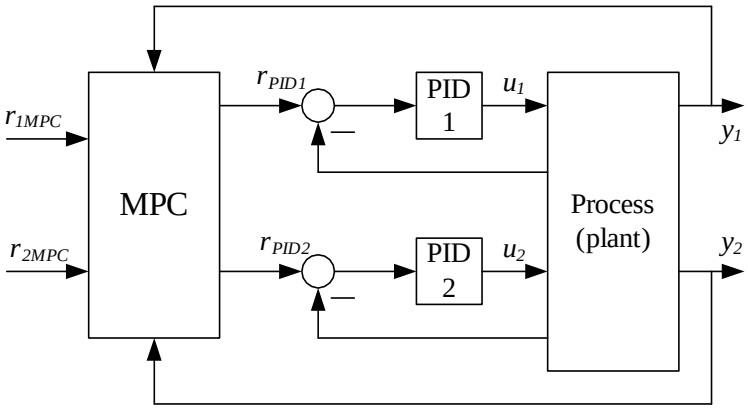

Figure 22.1:

Suggest concrete examples of r_{1MPC}, r_{PID_1}, and u_1 (select any application you want).

Chapter 23

Dead-time compensator (Smith predictor)

Exercise 23.1

1. Assume given a process with time-constant 1.2 min and dead-time 2 min. Will dead-compensation improve the disturbance compensation properties of the control system compared to using standard PID control?

   What about the reference tracking – will it be improved?

2. What are the answers if the time-constant is 0.4 min and the dead-time is 2 min?

Exercise 23.2

Given a process with the following transfer function from control variable to process measurement:

$$H_p(s) = \frac{2}{20s+1}\, e^{-50s} \tag{23.1}$$

Calculate the parameter values of an internal PI controller using Skogestad's tuning method from the specification that the closed loop time becomes 10 sec. (You can use parameter $c = 2$ in Skogestad's formula for T_i.)

Exercise 23.3

Dead-time compensators may improve control when the process has dead-time. But, even better control can be obtained by eliminating the dead-time!

Figure 23.1 shows a level control system of a wood-chip tank. The level

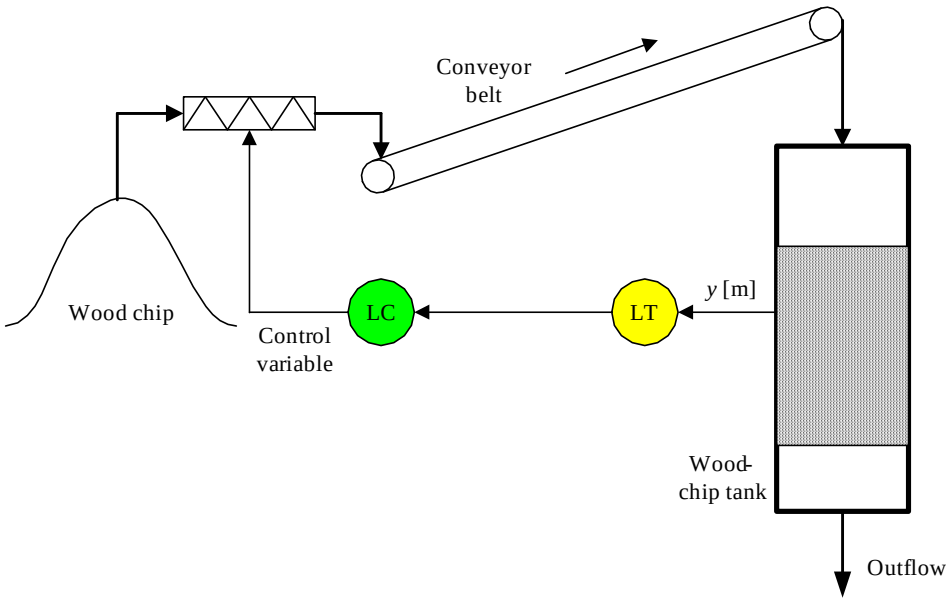

Figure 23.1:

controller adjusts the inflow to control the level in the tank, despite outflow variations.

1. In one specific paper pulp factory the level controller adjusts the speed of the inlet screw, while the conveyor belt is running with fixed speed. Explain why the control loop has dead-time.

2. The dead-time limits the speed of the level control system. Suggest an alternative way that the level controller can adjust the inflow so that the dead-time is eliminated!

Part II

SOLUTIONS

Solution to Exercise 1.1

Density ρ can be cancelled. (1.1) becomes

$$\underline{\underline{\dot{h}_1 = \overbrace{\frac{1}{A_1}\left(K_p u_1 - K_{v_1}\sqrt{\frac{\rho g h_1}{G}}\right)}^{f_1(\cdot)}}} \tag{23.2}$$

(1.2) becomes

$$\underline{\underline{\dot{h}_2 = \overbrace{\frac{1}{A_2}\left(K_{v_1}\sqrt{\frac{\rho g h_1}{G}} - K_{v_2} u_2\sqrt{\frac{\rho g h_2}{G}}\right)}^{f_2(\cdot)}}} \tag{23.3}$$

The measurement equations become

$$\underline{\underline{y_1 = \overbrace{h_1}^{g_1(\cdot)}}} \tag{23.4}$$

$$\underline{\underline{y_2 = \overbrace{h_2}^{g_2(\cdot)}}} \tag{23.5}$$

The state-space model is <u>nonlinear</u> due to the square root functions.

Solution to Exercise 1.2

Firstly, we isolate the first order derivatives on the left side, and list the variables in the proper order to prepare for the matrix-vector form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 3x_2 + 2u_1 + 4u_2 \\ y &= 5x_1 + 6x_2 + 7u_1 \end{aligned} \tag{23.6}$$

Finally,

$$\underline{\underline{\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{u}}} \tag{23.7}$$

and

$$\underline{\underline{y = \underbrace{\begin{bmatrix} 5 & 6 \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 7 & 0 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{u}}} \tag{23.8}$$

or, compactly:

$$\dot{x} = Ax + Bu \tag{23.9}$$

$$y = Cx + Du \tag{23.10}$$

Solution to Exercise 1.3

Linearization of the differential equations:

$$\begin{aligned}
\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{bmatrix} &= \left.\begin{bmatrix} \frac{\partial f_1}{\partial h_1} & \frac{\partial f_1}{\partial h_2} \\ \frac{\partial f_2}{\partial h_1} & \frac{\partial f_2}{\partial h_2} \end{bmatrix}\right|_0 \cdot \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} && (23.11)\\
&+ \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}\right|_0 \cdot \begin{bmatrix} \Delta u_1 \\ \Delta u_1 \end{bmatrix} && (23.12)\\
&= \underbrace{\left.\begin{bmatrix} -\frac{K_{v_1}}{A_1}\sqrt{\frac{\rho g}{G}}\frac{1}{2\sqrt{h_1}} & 0 \\ \frac{K_{v_1}}{A_2}\sqrt{\frac{\rho g}{G}}\frac{1}{2\sqrt{h_1}} & \frac{K_{v_2}u_2}{A_2}\sqrt{\frac{\rho g}{G}}\frac{1}{2\sqrt{h_2}} \end{bmatrix}\right|_0}_{A} \cdot \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} && (23.13)\\
&+ \underbrace{\left.\begin{bmatrix} \frac{K_p}{A_1} & 0 \\ 0 & \frac{K_{v_2}}{A_2}\sqrt{\frac{\rho g h_2}{G}} \end{bmatrix}\right|_0}_{B} \cdot \begin{bmatrix} \Delta u_1 \\ \Delta u_1 \end{bmatrix} && (23.14)
\end{aligned}$$

Linearization of the output equation:

$$\begin{aligned}
\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} &= \left.\begin{bmatrix} \frac{\partial g_1}{\partial h_1} & \frac{\partial g_1}{\partial h_2} \\ \frac{\partial g_2}{\partial h_1} & \frac{\partial g_2}{\partial h_2} \end{bmatrix}\right|_0 \cdot \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} && (23.15)\\
&+ \left.\begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} \end{bmatrix}\right|_0 \cdot \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} && (23.16)\\
&= \underbrace{\left.\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|_0}_{C} \cdot \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} && (23.17)\\
&+ \underbrace{\left.\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right|_0}_{D} \cdot \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} && (23.18)
\end{aligned}$$

Solution to Exercise 2.1

1. Figure 23.2 shows the signals with amplitudes and time-lag indicated. From the figure we see that the period of the input signal is

$$T_p = 10 \text{ sec} \tag{23.19}$$

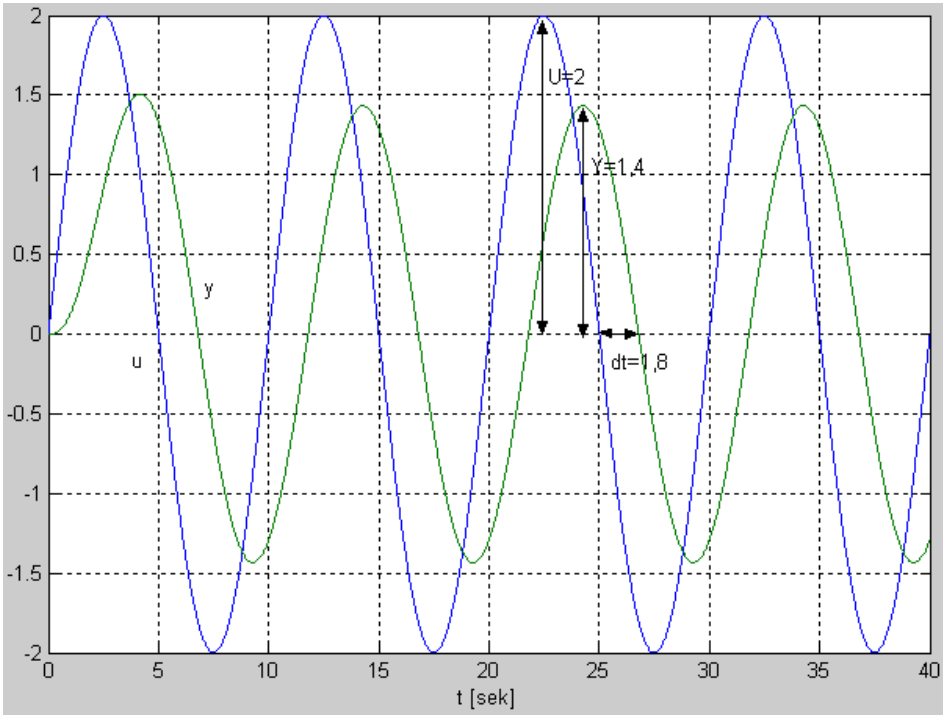

Figure 23.2:

which corresponds to the frequency

$$\underline{\underline{f_1}} = \frac{1}{T_p} = \frac{1}{10} = \underline{\underline{0.1 \text{ Hz}}} \tag{23.20}$$

or, alternatively,

$$\underline{\underline{\omega_1}} = 2\pi f_1 = 2 \cdot \pi \cdot 0.1 = \underline{\underline{0.63 \text{ rad/s}}} \tag{23.21}$$

2. The amplitude gain (at frequency f_1) is

$$\underline{\underline{A}} = \frac{Y}{U} = \frac{1.4}{2} = \underline{\underline{0.7}} = 20 \cdot \log_{10}(0.7) \text{ dB} = \underline{\underline{-3.1 \text{ dB}}} \tag{23.22}$$

The phase lag ϕ can be calculated by firstly measuring the time lag Δt between input $u(t)$ and output $y(t)$ and then calculating ϕ with

$$\phi = -\omega \Delta t \text{ [rad]} \tag{23.23}$$

From Figure 23.2 you can find

$$\Delta t = 1.8 \text{ s} \tag{23.24}$$

Hence,

$$\underline{\underline{\phi}} = -\omega\Delta t = -0.63 \cdot 1.8 = \underline{\underline{-1.13 \text{ rad}}} = -1.13 \cdot \frac{180}{\pi} = \underline{\underline{-65 \text{ deg}}} \tag{23.25}$$

Alternatively, we can calculate ϕ from the following ratio (360 degrees corresponds to one period):

$$\underline{\underline{\phi}} = -\frac{\Delta t}{T_p} \cdot 360 \text{ deg} = -\frac{1.8 \text{ s}}{10 \text{ s}} \cdot 360 \text{ deg} = \underline{\underline{-65 \text{ deg}}} \tag{23.26}$$

Solution to Exercise 2.2

The steady-state response is

$$y_s(t) = UA\sin(\omega t + \phi) \tag{23.27}$$

The amplitude of the input signal is $U = 0.8$. The amplitude gain A is read off from the upper curve in the Bode diagram at frequency $\omega = 1.0$ rad/s:

$$A = -3 \text{ dB} \tag{23.28}$$

which is

$$A = 10^{-3/20} = 0.71 \tag{23.29}$$

The phase lag ϕ at frequency $\omega = 1.0$ rad/s is read off from the lower curve in the Bode diagram:

$$\phi = -45 \text{ deg} = -45 \cdot \frac{\pi}{180} \text{ rad} = -0.79 \text{ rad} \tag{23.30}$$

Hence,

$$\underline{\underline{y_s(t) = 0.8 \cdot 0.71 \cdot \sin(1.0 \cdot t - 0.79) = 0.57\sin(t - 0.79)}} \tag{23.31}$$

Solution to Exercise 2.3

We set $s = j\omega$ in $H(s)$, and then turn the factors into *polar forms*, which we then combine to finally get a polar form of $H(j\omega)$:

$$\underline{\underline{H(j\omega)}} = \frac{K}{(1+T_1 j\omega)(1+T_2 j\omega)} e^{-j\omega\tau} \tag{23.32}$$

$$= \frac{K}{\left[\sqrt{1^2+(T_1\omega)^2} e^{j\arctan\left(\frac{T_1\omega}{1}\right)}\right]\left[\sqrt{1^2+(T_2\omega)^2} e^{j\arctan\left(\frac{T_2\omega}{1}\right)}\right]} e^{-j\omega\tau} \tag{23.33}$$

$$= \underbrace{\frac{K}{\sqrt{1+(T_1\omega)^2}\sqrt{1+(T_2\omega)^2}}}_{|H(j\omega)|} e^{j\underbrace{[-\arctan(T_1\omega)-\arctan(T_2\omega)-\omega\tau]}_{\arg H(j\omega)}} \tag{23.34}$$

So, we have

$$\underline{\underline{A(\omega) = |H(j\omega)| = \frac{K}{\sqrt{1+(T_1\omega)^2}\sqrt{1+(T_2\omega)^2}}}} \tag{23.35}$$

and

$$\underline{\underline{\phi(\omega) = \arg H(j\omega) = -\arctan(T_1\omega) - \arctan(T_2\omega)}} \tag{23.36}$$

Solution to Exercise 2.4

The bandwidth is

$$f_b \text{ [Hz]} = \frac{\omega_b \text{ [rad/s]}}{2\pi} = \frac{1}{2\pi}\cdot\frac{1}{RC} \tag{23.37}$$

With

$$C = 10^{-5} \text{ F} \tag{23.38}$$

we get

$$\underline{\underline{R}} = \frac{1}{2\pi f_b C} = \frac{1}{2\pi\cdot 100 \text{ Hz } \cdot 10^{-5} \text{ F}} = \underline{\underline{159\ \Omega}} \tag{23.39}$$

Solution to Exercise 3.1

Figure 23.3 shows a block diagram where the measurement noise n and a (lowpass) filter is included.

Solution to Exercise 3.2

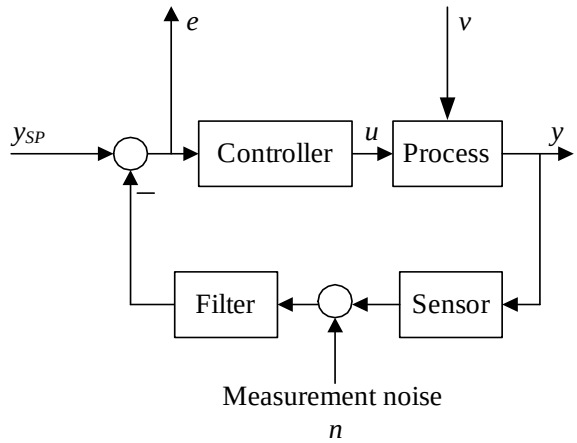

Figure 23.3:

$$\underline{\underline{L(s)}} = H_c(s)H_u(s)H_m(s) = \underline{\underline{K_p \frac{T_i s + 1}{T_i s} \frac{K_u}{T_u s + 1} e^{-\tau s} K_m}} \tag{23.40}$$

$$\begin{aligned}
\underline{\underline{S(s)}} &= \frac{1}{1 + L(s)} && (23.41)\\
&= \frac{1}{1 + K_p \frac{T_i s+1}{T_i s} \frac{K_u}{T_u s+1} e^{-\tau s} K_m} && (23.42)\\
&= \underline{\underline{\frac{T_i s \left(T_u s + 1\right)}{T_i s \left(T_u s + 1\right) + K_p K_u K_m \left(T_i s + 1\right) e^{-\tau s}}}} && (23.43)
\end{aligned}$$

$$\begin{aligned}
\underline{\underline{T(s)}} &= \frac{L(s)}{1 + L(s)} && (23.44)\\
&= \frac{K_p \frac{T_i s+1}{T_i s} \frac{K_u}{T_u s+1} e^{-\tau s} K_m}{1 + K_p \frac{T_i s+1}{T_i s} \frac{K_u}{T_u s+1} e^{-\tau s} K_m} && (23.45)\\
&= \underline{\underline{\frac{K_p K_u K_m \left(T_i s + 1\right) e^{-\tau s}}{T_i s \left(T_u s + 1\right) + K_p K_u K_m \left(T_i s + 1\right) e^{-\tau s}}}} && (23.46)
\end{aligned}$$

Solution to Exercises 3.3

1. From Figure 3.3 we read off

$$\underline{\underline{\omega_c = 2.5 \text{ rad/s}}} \tag{23.47}$$

$$\underline{\underline{\omega_t = 5.5 \text{ rad/s}}} \tag{23.48}$$

$$\underline{\underline{\omega_s = 0.55 \text{ rad/s}}} \tag{23.49}$$

(Hence, there is quite large difference between the different bandwidths in this example.)

2. The calculation from dB is made with the formula $x = 10^{x[\text{dB}]/20}$.

$$\underline{\underline{A_y}} = \overbrace{|T(j1\text{rad/s})|}^{=-0.8\text{dB}=0.91} A_{SP} = \underline{\underline{0.91 A_{SP}}} \tag{23.50}$$

$$\underline{\underline{A_e}} = \overbrace{|S(j1\text{rad/s})|}^{=-8\text{dB}=0.40} A_{SP} = \underline{\underline{0.40 A_{SP}}} \tag{23.51}$$

3.

$$\underline{\underline{A_{y_{\text{CL}}}}} = \overbrace{|S(j1\text{rad/s})|}^{=-8\text{dB}=0{,}40} A_{y_{\text{OL}}} = 0.40 \cdot 0.5 = \underline{\underline{0.2}} \tag{23.52}$$

4.

$$\underline{\underline{T_r}} \approx \frac{2}{\omega_t} = \frac{2}{5.5} = \underline{\underline{0.36 \text{ s}}} \tag{23.53}$$

Solution to 3.4

1. In Figure 3.5, which applies to the non-controlled system, we read off that the amplitude gain at frequency 0.1 rad/s is

$$\underline{\underline{G_{\text{NC}}(0.1) \approx 0 \text{ dB} = 1}} \tag{23.54}$$

2. In Figure 3.6, which applies to the controlled system, we read off that the amplitude gain at frequency 0.1 rad/s is

$$\underline{\underline{G_{\text{C}}(0.1) \approx -32 \text{ dB} = 0.025}} \tag{23.55}$$

Hence, with control the response in T due to T_{in} is about <u>40 times less</u> than the response in the non-controlled system.

Solution to Exercise 4.1

The Laplace transform of the impulse response is

$$h(t) = y(s) = H(s)\underbrace{u(s)}_{=1} = H(s) = \frac{1}{s+1} \tag{23.56}$$

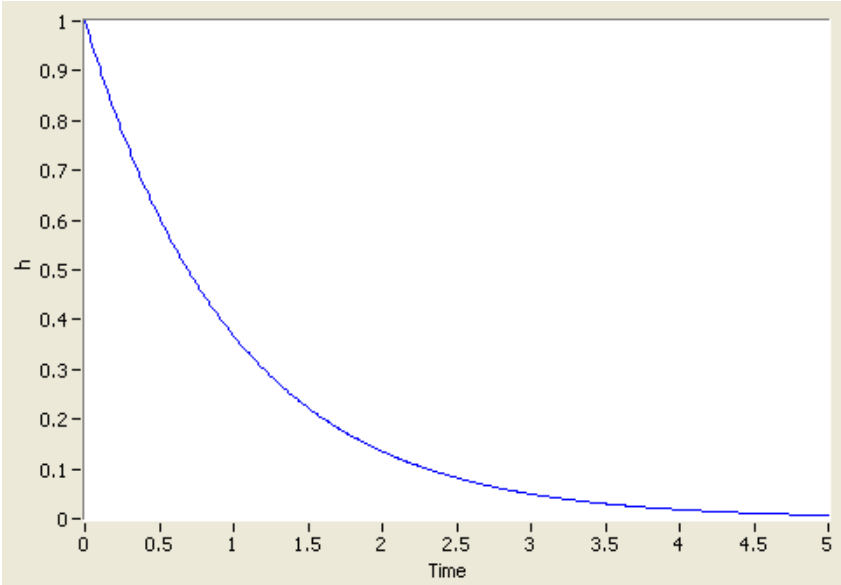

Figure 23.4:

Using (4.2) with $k = 1$ and $T = 1$ we get

$$h(t) = \frac{ke^{-t/T}}{T} = e^{-t} \tag{23.57}$$

Figure 23.4 shows $h(t)$. Since $h(t)$ goes to zero as time goes to infinity, the system is asymtptotically stable.

Solution to Exercise 4.2

The transfer function

$$H_1(s) = \frac{1}{s+1} \tag{23.58}$$

is asymptotically stable since the pole $p = -1$ is in the left half plane.

The transfer function

$$H_2(s) = \frac{1-s}{1+s} \tag{23.59}$$

is asymptotically stable since the pole $p = -1$ is in the left half plane. The value of the zero, which is 1, does not determine the stability.

The transfer function

$$H_3(s) = \frac{1}{1-s} \tag{23.60}$$

is <u>unstable</u> since the pole $p = 1$ is in the right half plane.

The transfer function

$$H_4(s) = \frac{1}{(s+1)(s-1)} \tag{23.61}$$

is <u>unstable</u> since one of the poles, $p_1 = 1$, is in the right half plane.

The transfer function

$$H_5(s) = \frac{1}{s} \tag{23.62}$$

is <u>marginally stable</u> since the pole $p = 0$ is in origin, which is on the imaginary axis, and this pole is single (there are no multiple poles).

The transfer function

$$H_6(s) = \frac{1}{s^3} \tag{23.63}$$

is <u>unstable</u> since there are multiple poles, $p_{1,2,3} = 0$, on the imaginary axis.

The transfer function

$$H_7(s) = \frac{e^{-s}}{s+1} \tag{23.64}$$

is <u>asymptotically stable</u> since the pole $p = -1$ is in the left half plane.

The transfer function

$$H_8(s) = -\frac{1}{s+1} \tag{23.65}$$

is <u>asymptotically stable</u> since the pole $p = -1$ is in the left half plane.

The transfer function

$$H_9(s) = \frac{1}{s^2+s+1} \tag{23.66}$$

is <u>asymptotically stable</u> since both the poles

$$p_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{4 \cdot 1} = \frac{-1 \pm j\sqrt{3}}{4} \tag{23.67}$$

are in the left half plane.

The transfer function

$$H_{10}(s) = \frac{1}{s^2+1} \tag{23.68}$$

is <u>marginally stable</u> since the poles

$$p_{1,2} = \pm j \tag{23.69}$$

are on the imaginary axis and they are single (not multiple).

The transfer function

$$H_{11}(s) = \frac{1}{(s+1)s} \tag{23.70}$$

has poles

$$p_{1,2} = 0, -1 \tag{23.71}$$

One pole is on the imaginary axis, and the other is in the left half plane. The system is <u>marginally stable</u>.

Solution to Exercise 4.3

The stability property is determined by the system eigenvalues, which are the roots of the characteristic equation:

$$\det(sI - A) = \det\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\right) \tag{23.72}$$

$$= \det\left(\begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}\right) \tag{23.73}$$

$$= s\,(s+2) + 0 \tag{23.74}$$

The roots are

$$s_{1,2} = 0, \ -2 \tag{23.75}$$

One pole is in the origin, and the other pole is in the left half plane. Thererfore, the system is <u>marginally stable</u>.

Solution to Exercise 5.1

1. $H_{pm}(s)$ has pole equal to 0. Consequently the system is <u>marginally stable</u>.
2. The transfer function of the controller is

   $$H_c(s) = K_p \tag{23.76}$$

   The loop transfer function is

   $$L(s) = H_c(s)H_{pm}(s) = \frac{K_p}{s} = \frac{n_0(s)}{d_0(s)} \tag{23.77}$$

   The characteristic polynomial is

   $$c(s) = d_0(s) + n_0(s) = s + K_p \tag{23.78}$$

   The pole of the control system is the root of (23.78):

   $$p = -K_p \tag{23.79}$$

   The control system is asymptotically stable with $\mathrm{Re}(p) < 0$, i.e.

   $$\underline{\underline{K_p > 0}} \tag{23.80}$$

3. Sure!

Solution to Exercise 5.2

We start by determining the stability property with $K_p = 0.4$. The number of right half plane poles of the control system is

$$P_{CL} = \frac{\arg[1 + H_0(s)]}{360^\circ} + P_{OL} \tag{23.81}$$

To determine P_{CL} we need to know $\arg L$ and P_{OL}. We have

$$L(s) = \frac{K_p}{(s+1)^3 s} \tag{23.82}$$

which does not have any poles in the right half plane (the pole in the origin belongs to the left half plane in this context). Hence,

$$P_{OL} = 0 \tag{23.83}$$

$\arg L$ is found from the Nyquist diagram shown in Figure 5.2. $L(j\omega)$ does not encircle the critical point, and therefore the vector L has zero net change of angle, and hence $\arg L = 0$. So, (23.81) becomes

$$P_{CL} = \frac{0}{360^\circ} + 0 = 0 \tag{23.84}$$

Consequently, the control system is asymptotically stable with $K_p = 0.4$.

From Figure 5.2 we see that the L curve with $K_p = 0.4$ passes through the negative real axis at -0.45. This implies that if K_p is increased by a factor of $1/0.45 = 2.22$, or in other words: if K_p is increased from 0.4 to $0.4 \cdot 2.22 = 0.89$, the L curve will pass through the critical point.

Consequently, the control system is <u>marginally stable with $K_p = 0.89$</u>.

From the results above we can conclude that the control system is <u>asymptotically stable with (positive) $K_p < 0.89$</u>.

If $K_p > 0.89$, the L curve encircles the critical point and $\arg L = 720^\circ$, giving

$$P_{CL} = \frac{720^\circ}{360^\circ} + 0 = 2 \tag{23.85}$$

Therefore, the control system is <u>unstable with $K_p > 0.89$</u>, and it has <u>two poles in the right half plane</u>.

The control system is <u>unstable also with $K_p < 0$</u>. In this case $\arg L = 360^\circ$, and $P_{CL} = 1$, and the control system has <u>one pole in the right half plane</u>.

Solution 5.3

1. The open loop system is unstable because $L(s)$ has a pole in the right half plane. (The pole is $p = 1$.)

2. K must be halved to make the L curve pass through the critical point:
$$K_{critical} = \frac{2}{2} = 1 \tag{23.86}$$
Since the open loop system has one pole in the right half plane, $P_{OL} = 1$ in the Nyquist criterion. To make $P_{CL} = 0$ (asymptotically stable closed loop system) $\arg L$ must be 360°, which implies that the critical point must be encircled once. This is achieved with
$$\underline{\underline{K > K_{critical} = 1}} \tag{23.87}$$
The pole of the closed loop is
$$p = 1 - K \tag{23.88}$$
This pole is in the left half plane with $K > 1$, which confirms the result of the Nyquist stability criterion above.

Solution to Exercise 5.4

The Nyquist curve passes through the critical point with the following values of K:
$$K = 1/0.8 = 1.25 \tag{23.89}$$
$$K = 1/0.4 = 2.5 \tag{23.90}$$
$$K = 1/0.2 = 5 \tag{23.91}$$

The control system is asymptotically stable with
$$\underline{\underline{0 < K < 1.25 \text{ and } 2.5 < K < 5}} \tag{23.92}$$

Solution to Exercise 5.5

See Figure 23.5. As indicated in the figure,
$$0.4 = |1 + L|_{\min} = \frac{1}{|S|_{\max}} \tag{23.93}$$
which gives
$$\underline{\underline{|S|_{\max}}} = \frac{1}{0.4} = \underline{\underline{2.5 = 8.0 \text{ dB}}} \tag{23.94}$$

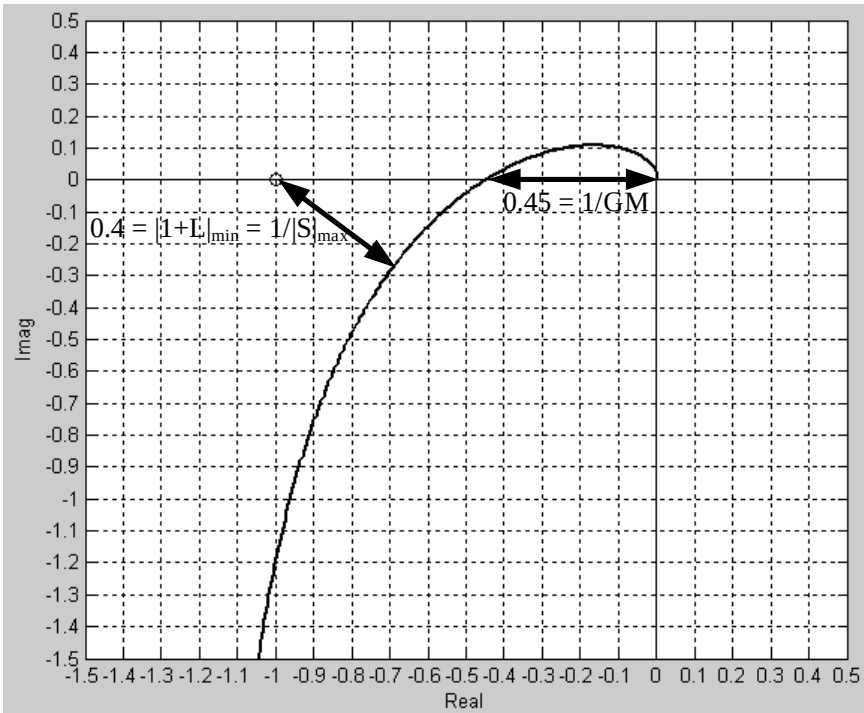

Figure 23.5:

Furthermore,

$$\frac{1}{GM} = 0.45 \tag{23.95}$$

which gives

$$\underline{\underline{GM}} = \frac{1}{0.45} = \underline{\underline{2.22 = 6.9 \text{ dB}}} \tag{23.96}$$

Solution to Exercise 5.6

From Figure 5.6 we read off

$$|S(j\omega)|_{\max} = 7 \text{ dB} \tag{23.97}$$

which is in the "reasonable" range of

$$3.5 \text{ dB} \leq |S(j\omega)|_{\max} \leq 9.5 \text{ dB} \tag{23.98}$$

Solution to Exercise 5.7

1. See Figure 23.6. From the Bode diagram:

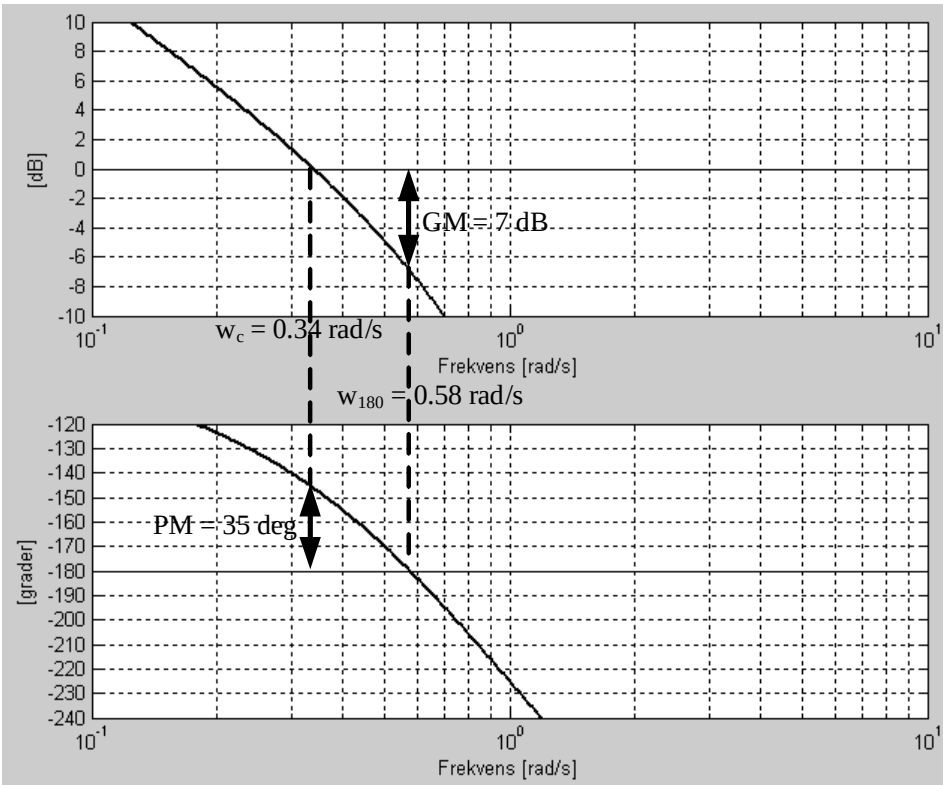

Figure 23.6:

$$\underline{\underline{GM = 7 \text{ dB} = 2.2}} \tag{23.99}$$

$$\underline{\underline{PM = 35^\circ}} \tag{23.100}$$

$$\underline{\underline{\omega_c = 0.34 \text{ rad/s}}} \tag{23.101}$$

$$\underline{\underline{\omega_{180} = 0.58 \text{ rad/s}}} \tag{23.102}$$

2. An increase of the loop gain by a factor of

$$\underline{\underline{GM = 2.2}} \tag{23.103}$$

will bring the system to the stability limit.

The period is

$$\underline{\underline{T_p = \frac{2\pi}{\omega_{180}} = \frac{2\pi}{0.58} = 10.8 \text{ s}}} \tag{23.104}$$

Solution 5.8

When the control system is marginally stable, the phase margin PM is zero. According to Eq. (5.44) in the text-book the maximum change of the time delay that is allowed before the system is marginally stable is

$$\Delta\tau = \frac{PM}{\omega_c} \cdot \frac{\pi}{180^\circ} = \frac{45^\circ}{0.2} \cdot \frac{\pi}{180^\circ} = 3.9 \text{ min} \tag{23.105}$$

Since the time delay *before* the change is 4.2 min, the total value of the time delay at marginally stability is

$$\underline{\underline{\tau}} = 4.2 + 3.9 = \underline{\underline{8.1 \text{ min}}} \tag{23.106}$$

Solution to Exercise 5.9

For $H_1(s)$ the loop transfer function with a P controller is

$$L_1(s) = K_p H_1(s) = K_p \frac{K}{s} \tag{23.107}$$

The phase (angle) of $L_1(j\omega)$ converges to -90° as the frequency goes to infinity. Therefore, the closed loop system will not become marginally stable with any controller gain, and hence, the Ziegler-Nichols' method can not be used.

For $H_2(s)$ the loop transfer function with a P controller is

$$L_2(s) = K_p H_2(s) = K_p \frac{K}{Ts+1} \tag{23.108}$$

The phase (angle) of $L_2(j\omega)$ converges to -90° as the frequency goes to infinity. Therefore, the closed loop system will not become marginally stable with any controller gain, and hence, the Ziegler-Nichols' method can not be used.

For $H_3(s)$ the loop transfer function with a P controller is

$$L_3(s) = K_p H_3(s) = K_p \frac{K{\omega_0}^2}{s^2 + 2\zeta\omega_0 + {\omega_0}^2} \tag{23.109}$$

The phase (angle) of $L_3(j\omega)$ converges to -180° as the frequency goes to infinity. Therefore, the closed loop system will not become marginally stable with any limited controller gain, and hence, the Ziegler-Nichols' method can not be used.

Solution to Exercise 6.1

1.

$$\underline{\underline{x_d(t_k) = 2t_k}} \tag{23.110}$$

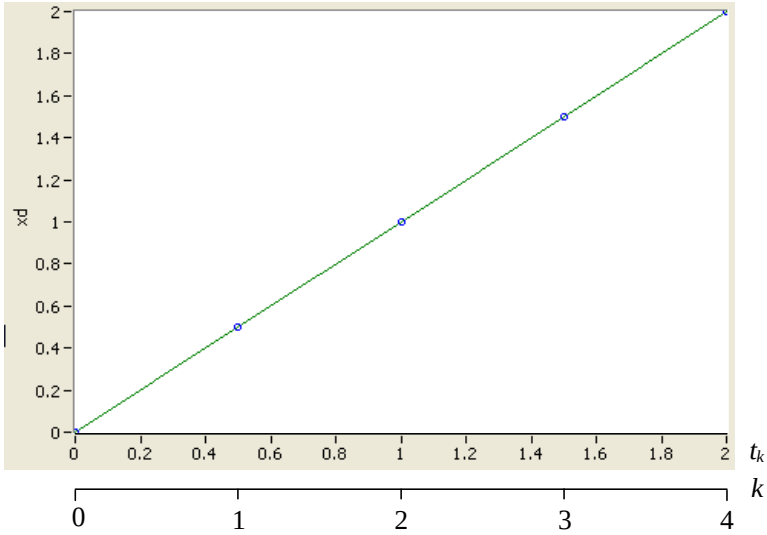

Figure 23.7:

$$\underline{\underline{x_d = \{0, 0.5, 1.0, 1.5, 2.0\}}} \tag{23.111}$$

Figure 23.7 shows x_d with $T_s = 0.5$ s.

2.

$$\underline{\underline{x_d(t_k) = 2t_k}} \tag{23.112}$$

$$\underline{\underline{x_d = \{0, 0.1, 0.2, 0.3, \ldots, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0\}}} \tag{23.113}$$

Figure 23.8 shows x_d with $T_s = 0.1$ s.

Solution to Exercise 7.1

Each of the time indexes is reduced by 3, giving

$$\underline{\underline{y(k) + ay(k-2) = b_1 u(k-1) + b_0 u(k-3)}} \tag{23.114}$$

Solution to Exercise 7.2

See Figure 23.9.

Solution to Exercise 7.3

1. Let us assume that

$$u = U \text{ (constant)} \tag{23.115}$$

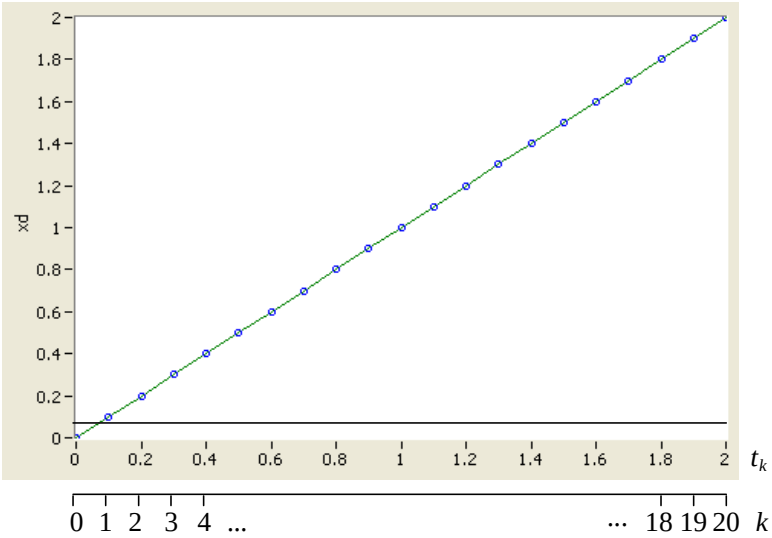

Figure 23.8:

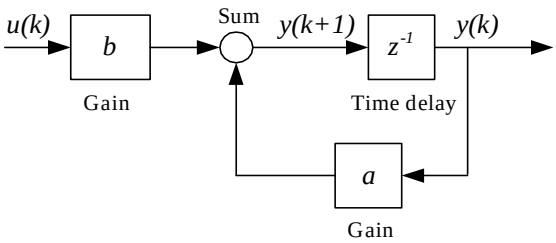

Figure 23.9:

In steady-state,

$$\underline{\underline{y_s}} = \frac{1}{3}\left(U + U + U\right) = \underline{\underline{U}}$$

Hence, the filter lets a constant input pass unchanged, in steady-state.

2. The ramp response is

$$
\begin{aligned}
\underline{\underline{y(0)}} &= \frac{1}{3}\left[u(0)+u(-1)+u(-2)\right] = \frac{1}{3}\left[0+0+0\right] = \underline{\underline{0}} \\
\underline{\underline{y(1)}} &= \frac{1}{3}\left[u(1)+u(0)+u(-1)\right] = \frac{1}{3}\left[0.5+0+0\right] = \frac{1}{6} = \underline{\underline{0.17}} \\
\underline{\underline{y(2)}} &= \frac{1}{3}\left[u(2)+u(1)+u(0)\right] = \frac{1}{3}\left[1.0+0.5+0\right] = \underline{\underline{0.5}} \\
\underline{\underline{y(3)}} &= \frac{1}{3}\left[u(3)+u(2)+u(1)\right] = \frac{1}{3}\left[1.5+1.0+0.5\right] = \underline{\underline{1}} \\
\underline{\underline{y(4)}} &= \frac{1}{3}\left[u(4)+u(3)+u(2)\right] = \frac{1}{3}\left[2.0+1.5+1.0\right] = \underline{\underline{1.5}}
\end{aligned}
$$

The general response:

$$
\begin{aligned}
\underline{\underline{y(k)}} &= \frac{1}{3}\left[u(k)+u(k-1)+u(k-2)\right] \\
&= \frac{1}{3}\left[0.5\cdot k+0.5\cdot(k-1)+0.5\cdot(k-2)\right] \\
&= 0.5\cdot k+\frac{1}{3}\left(0-0.5-1.0\right) \\
&= \underline{\underline{0.5\cdot k-0.5}} \\
&= u(k)-0.5
\end{aligned}
$$

Hence, there is a $\underline{\underline{\text{constant difference equal to } -0.5}}$ between the output and the input in steady-state.

Solution to Exercise 8.1

See Figure 23.10.

Solution to Exercise 8.2

Firstly, we write the model with the time-derivatives alone on the left hand side:

$$\dot{h}_1 = \frac{1}{A_1}\left(K_p u_1 - K_{v_1}\sqrt{\frac{\rho g h_1}{G}}\right) \tag{23.116}$$

$$\dot{h}_2 = \frac{1}{A_2}\left(K_{v_1}\sqrt{\frac{\rho g h_1}{G}} - K_{v_2} u_2\sqrt{\frac{\rho g h_2}{G}}\right) \tag{23.117}$$

(The model is now actually on state space model form.)

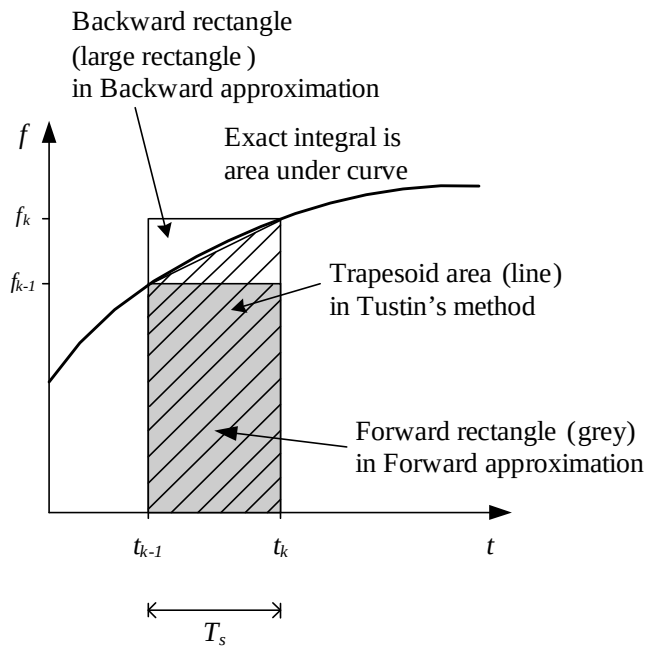

Figure 23.10:

Next, we apply the Forward differentiation approximation:

$$\dot{h}_1(t_k) \approx \frac{h_1(t_{k+1}) - h_1(t_k)}{T_s} = \frac{1}{A_1}\left(K_p u_1(t_k) - K_{v_1}\sqrt{\frac{\rho g h_1(t_k)}{G}}\right) \tag{23.118}$$

$$\dot{h}_2(t_k) \approx \frac{h_2(t_{k+1}) - h_2(t_k)}{T_s} = \frac{1}{A_2}\left(K_{v_1}\sqrt{\frac{\rho g h_1(t_k)}{G}} - K_{v_2} u_2\sqrt{\frac{\rho g h_2(t_k)}{G}}\right) \tag{23.119}$$

Solving for $h_1(t_{k+1})$ and $h_2(t_{k+1})$, respectively:

$$\underline{\underline{h_1(t_{k+1}) = h_1(t_k) + \frac{T_s}{A_1}\left(K_p u_1(t_k) - K_{v_1}\sqrt{\frac{\rho g h_1(t_k)}{G}}\right)}} \tag{23.120}$$

$$\underline{\underline{h_2(t_{k+1}) = h_2(t_k) + \frac{T_s}{A_2}\left(K_{v_1}\sqrt{\frac{\rho g h_1(t_k)}{G}} - K_{v_2} u_2\sqrt{\frac{\rho g h_2(t_k)}{G}}\right)}} \tag{23.121}$$

It is better to apply Forward discretization than Backward discretization in this example because the model is nonlinear (the square root function). The nonlinearity makes it difficult to solve for the levels h_1 and h_2 in a model created with Backward discretization, while solving for h_1 and h_2 is

straightforward in a model created with Forward discretization (this is demonstrated in the text-bok).

Solution to Exercise 8.3

Cross-multiplying the transfer function model:

$$(T_f s + 1)\, y(s) = T_f s u(s) \tag{23.122}$$

or

$$T_f s y(s) + y(s) = T_f s u(s) \tag{23.123}$$

Taking inverse Laplace tranform:

$$T_f \dot{y}(t) + y(t) = T_f \dot{u}(t) \tag{23.124}$$

Discrete time:

$$T_f \dot{y}(t_k) + y(t_k) = T_f \dot{u}(t_k) \tag{23.125}$$

Backward discretization:

$$\dot{y}(t_k) \approx \frac{y(t_k) - y(t_{k-1})}{T_s} \tag{23.126}$$

and

$$\dot{u}(t_k) \approx \frac{u(t_k) - u(t_{k-1})}{T_s} \tag{23.127}$$

Inserting these approximations into (23.125):

$$T_f \frac{y(t_k) - y(t_{k-1})}{T_s} + y(t_k) = T_f \frac{u(t_k) - u(t_{k-1})}{T_s} \tag{23.128}$$

Solving for $y(t_k)$:

$$\underline{\underline{y(t_k) = \frac{T_f}{T_s + T_f} y(t_{k-1}) + \frac{T_f}{T_s + T_f} \left[u(t_k) - u(t_{k-1})\right]}} \tag{23.129}$$

Solution to Exercise 8.4

1. Discretizing the nominal (manual) control variable u_0:

$$\underline{u_0(t_k) = u_0} \tag{23.130}$$

   Discretizing the P term:

$$\underline{u_p(t_k) = K_p e(t_k)} \tag{23.131}$$

Discretizing the I term: The continuous-time I term can be written

$$u_i(t_k) = \frac{K_p}{T_i} \int_0^{t_k} e(t)\, dt \tag{23.132}$$

Taking the time-derivative of (23.132):

$$\dot{u}_i(t_k) = \frac{K_p}{T_i} e(t_k) \tag{23.133}$$

Applying Backward differentiation approximation to the time-derivative:

$$\dot{u}_i(t_k) \approx \frac{u_i(t_k) - u_i(t_{k-1})}{T_s} = \frac{K_p}{T_i} e(t_k) \tag{23.134}$$

Solving for $u_i(t_k)$:

$$\underline{u_i(t_k) = u_i(t_{k-1}) + \frac{K_p T_s}{T_i} e(t_k)} \tag{23.135}$$

The total discrete-time PI control function is then

$$\underline{\underline{u(t_k) = u_0 + u_p(t_k) + u_i(t_k)}} \tag{23.136}$$

2. Integral anti windup can be implemented as follows:

   - Calculate an intermediate value of the control variable $u(t_k)$ according to (23.136), but do not send this value to the actuator (which controls the process).
   - Check if the intermediate $u(t_k)$ is greater than the maximum value $u_{\max}$ (typically 100%) or less than the minimum value $u_{\min}$ (typically 0%). If $u(t_k)$ is exceeding one of these limits, calculate the I term $u_i(t_k)$ once more, but now with

     $$u_i(t_k) = u_i(t_{k-1}) \tag{23.137}$$

     which implies that the I term is fixed, and calculate $u(t_k)$ once more according to (23.136) using $u_i(t_k)$ given by (23.137).
   - Send $u(t_k)$ to the actuator.

Solution to Exercise 8.5

1. Setting $T_i = \infty$ and $T_d = 0$ in the PID controller gives

$$\underline{\underline{u(t_k) = u(t_{k-1}) + [u_0(t_k) - u_0(t_{k-1})] + K_p\,[e(t_k) - e(t_{k-1})]}} \tag{23.138}$$

2. In steady state (23.138) becomes

$$\begin{aligned} u(t_k) &= u(t_{k-1}) + \underbrace{[u_0(t_k) - u_0(t_{k-1})]}_{=0} + K_p \underbrace{[e_s(t_k) - e_s(t_{k-1})]}_{=0} \\ &= u(t_{k-1}) + 0 + K_p \cdot 0 \end{aligned} \tag{23.139}$$

   So, the control signal, u, will not change, even if the controller gain is increased. In other words: The P controller seems "dead". This is an inherent drawback of a discrete-time P controller based on the algorithm (23.138).

3. The I controller is

$$u(t_k) = u(t_{k-1}) + \underbrace{\frac{K_p T_s}{T_i} e(t_k)}_{\Delta u} \tag{23.140}$$

   In (23.140) the control signal will change – or in other words: the control increment or change, Δu, will be nonzero – only as long as the control error is nonzero. So, the control signal will keep on changing until the control error has become zero.

Solution to Exercise 8.6

The time-step should be selected according to

$$\underline{\underline{T_s \leq \frac{T_r}{5} = \frac{50}{5} = 10 \text{ sec}}} \tag{23.141}$$

So, $T_s = 0.1$ sec is definitely <u>ok</u>!

Solution to Exercise 8.7

1. Case B: $\underline{\underline{T_s = 0.5 \text{ sec}}}$

   Case C: $\underline{\underline{T_s = 1 \text{ sec}}}$

2. Increased T_s gives increased time-delay in the AD-converter and consequently in control loop, causing reduced stability in the control loop.

Solution to Exercise 9.1

Solving (9.3) – (9.4) for $h_1(t_{k+1})$ and $h_2(t_{k+1})$, respectively, gives the discrete-time state space model:

$$\underline{\underline{h_1(t_{k+1}) = \overbrace{h_1(t_k) + \frac{T_s}{A_1}\left(K_p u_1(t_k) - K_{v_1}\sqrt{\frac{\rho g h_1(t_k)}{G}}\right)}^{f_1}}} \tag{23.142}$$

$$\underline{\underline{h_2(t_{k+1}) = \overbrace{h_2(t_k) + \frac{T_s}{A_2}\left(K_{v_1}\sqrt{\frac{\rho g h_1(t_k)}{G}} - K_{v_2} u_2(t_k)\sqrt{\frac{\rho g h_2(t_k)}{G}}\right)}^{f_2}}} \tag{23.143}$$

The outputs variables are

$$\underline{\underline{y_1(t_k) = h_1(t_k)}} \tag{23.144}$$

$$\underline{\underline{y_2(t_k) = h_2(t_k)}} \tag{23.145}$$

This state-space model is <u>nonlinear</u> due to the square root functions in which the state variables are arguments.

Solution to Exercise 9.2

$$\underline{\underline{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} -0.5 & 0 \\ -3 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{B} u(k)}} \tag{23.146}$$

$$\underline{\underline{y(k) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \underbrace{[4]}_{D} u(k)}} \tag{23.147}$$

Solution to Exercise 9.3

By applying Backward differentiation approximation to the time-derivatives, we get the following discrete-time model:

$$\dot{h}_1(t_k) \approx \frac{h_1(t_k) - h_1(t_{k-1})}{T_s} = \frac{1}{A_1}\left(K_p u_1(t_k) - K_{v_1}\sqrt{\frac{\rho g h_1(t_k)}{G}}\right) \tag{23.148}$$

$$\dot{h}_2(t_k) \approx \frac{h_2(t_k) - h_2(t_{k-1})}{T_s} = \frac{1}{A_2}\left(K_{v_1}\sqrt{\frac{\rho g h_1(t_k)}{G}} - K_{v_2} u_2(t_k)\sqrt{\frac{\rho g h_2(t_k)}{G}}\right) \tag{23.149}$$

By increasing the time-index with one we get

$$\frac{h_1\left(t_{k+1}\right)-h_1\left(t_k\right)}{T_s}=\frac{1}{A_1}\left(K_p u_1\left(t_{k+1}\right)-K_{v_1}\sqrt{\frac{\rho g h_1\left(t_{k+1}\right)}{G}}\right) \tag{23.150}$$

$$\frac{h_2\left(t_{k+1}\right)-h_2\left(t_k\right)}{T_s}=\frac{1}{A_2}\left(K_{v_1}\sqrt{\frac{\rho g h_1\left(t_{k+1}\right)}{G}}-K_{v_2} u_2\left(t_{k+1}\right)\sqrt{\frac{\rho g h_2\left(t_{k+1}\right)}{G}}\right) \tag{23.151}$$

which can *not* be written on the following standard state-space model form:

$$h_1\left(t_{k+1}\right)=f_1\left[h_1(t_k), h_2(t_k), \cdots\right] \tag{23.152}$$

$$h_2\left(t_{k+1}\right)=f_2\left[h_1(t_k), h_2(t_k), \cdots\right] \tag{23.153}$$

because $h_1\left(t_{k+1}\right)$ and $h_2\left(t_{k+1}\right)$ appear in square root functions on the right sides.

Solution to Exercise 9.4

1. <u>No, because the continuous-time state-space model is nonlinear.</u>
2. <u>Yes, because the model is linear.</u>

Solution to Exercise 10.1

$$\underline{\underline{\mathcal{Z}\left\{y(k)\right\}=y(z)}} = \sum_{k=0}^{\infty} y(k) z^{-k}=\sum_{k=0}^{\infty} \delta(k) z^{-k} \tag{23.154}$$

$$= 1 \cdot z^{-0}+0 \cdot z^{-1}+0 \cdot z^{-2}+\cdots+0 \cdot z^{-n}+\cdots \tag{23.155}$$

$$= \underline{\underline{1}} \tag{23.156}$$

Solution to Exercise 10.2

The $\mathcal{Z}$-transform of the right side of (10.1) becomes

$$\mathcal{Z}\{A S(k)+B S_2(k)\}=\mathcal{Z}\{(A+B) S(k)\}=(A+B) \frac{z}{z-1} \tag{23.157}$$

The left side of (10.1) becomes

$$A \cdot \mathcal{Z}\{S(k)\}+B \cdot \mathcal{Z}\{S(k)\}=A \frac{z}{z-1}+B \frac{z}{z-1}=(A+B) \frac{z}{z-1} \tag{23.158}$$

Hence, the $\mathcal{Z}$-transform of the right side of (10.1) is equal to the left side of (10.1), and consequently the linear property holds.

Solution to Exercise 10.3

The signal is

$$y(k) = 4 \cdot S(k-2) \tag{23.159}$$

Using the Linearity property and the Time-delay property of the $\mathcal{Z}$-transform, we get

$$\underline{\underline{\mathcal{Z}\{y(k)\} = y(z)}} = 4z^{-2}\mathcal{Z}\{S(k)\} = \underline{\underline{4z^{-2}\frac{z}{z-1} = \frac{4}{z^2 - z}}} \tag{23.160}$$

Solution to Exercise 10.4

Using the $\mathcal{Z}$-transform pair denoted "Time exponential" we get

$$\underline{\underline{y(k) = 0.5^k}} \tag{23.161}$$

Solution to Exercise 11.1

z transformation gives

$$y(z) = \frac{1}{3}\left[u(z) + z^{-1}u(z) + z^{-2}u(z)\right] = \frac{1}{3}\left[1 + z^{-1} + z^{-2}\right]u(z) \tag{23.162}$$

which gives the transfer function

$$\underline{\underline{\frac{y(z)}{u(z)} = \frac{1}{3}\left[1 + z^{-1} + z^{-2}\right]}} \tag{23.163}$$

Solution to Exercise 11.2

Cross-multiplication gives

$$\left[bz^2 + cz + d\right]y(z) = au(z)$$

or

$$bz^2y(z) + czy(z) + dy(z) = au(z) \tag{23.164}$$

which inverse-transformed gives this difference equation:

$$\underline{\underline{by(k+2) + cy(k+1) + dy(k) = au(k)}}$$

Solution to Exercise 11.3

The $\mathcal{Z}$-transform of y becomes

$$y(z) = H(z)u(z) = \frac{z}{z-1} \cdot A \tag{23.165}$$

$y(k)$ is given by the inverse transform of $y(z)$:

$$\underline{\underline{y(k)}} = \mathcal{Z}^{-1}\{y(z)\} = \mathcal{Z}^{-1}\{A\frac{z}{z-1}\} = \underline{\underline{A}} \tag{23.166}$$

(which is a step of amplitude A at time zero).

Solution to Exercise 11.4

1. The static transfer function is

$$\underline{\underline{H_s}} = H(z=1) = \frac{1}{3}\left[1 + 1^{-1} + 1^{-2}\right] = \underline{\underline{1}} \tag{23.167}$$

2. The steady-state filter output is

$$\underline{\underline{y_s}} = H_s U = 1 \cdot U = \underline{\underline{U}} \tag{23.168}$$

Solution to Exercise 11.5

It is convenient to start by rewriting the transfer function as follows:

$$H(z) = \frac{z^{-2}b + z^{-1}}{1 - az^{-1}} \cdot \frac{z^2}{z^2} = \frac{z+b}{z^2 - az} = \frac{z+b}{(z-a)\,z} \tag{23.169}$$

Thus, the zero z is

$$\underline{\underline{z = -b}} \tag{23.170}$$

and the poles p_i are

$$\underline{\underline{p_1 = a;\ p_2 = 0}} \tag{23.171}$$

Solution to Exercise 11.6

(11.4) can be written

$$sy(s) = u(s) \tag{23.172}$$

Inverse Laplace transform gives

$$\dot{y}(t) = u(t) \tag{23.173}$$

Applying Backward discretization to the time-derivative and introducing discrete time notation:

$$\dot{y}(t_k) \approx \frac{y(t_k) - y(t_{k-1})}{T_s} = u(t_k) \tag{23.174}$$

Solving for $y(t_k)$:

$$y(t_k) = y(t_{k-1}) + T_s u(t_k) \tag{23.175}$$

Taking the $\mathcal{Z}$-transform:

$$y(z) = z^{-1} y(z) + T_s u(z) \tag{23.176}$$

The transfer function becomes

$$\underline{\underline{H_{\text{disc}}(z) = \frac{y(z)}{u(z)} = \frac{T_s}{1 - z^{-1}} = \frac{zT_s}{z-1}}} \tag{23.177}$$

Solution to Exercise 12.1

The amplitude gain function is

$$\underline{\underline{A(\omega)}} = |H(e^{j\omega T_s})| = \left| \frac{1}{e^{j\omega T_s}} \right| = \frac{1}{|e^{j\omega T_s}|} = \frac{1}{1} = \underline{\underline{1}} \tag{23.178}$$

The phase lag function is

$$\underline{\underline{\phi(\omega)}} = \arg H(e^{j\omega T_s}) = \arg \frac{1}{e^{j\omega T_s}} = \arg e^{-j\omega T_s} = \underline{\underline{-\omega T_s = -\omega \cdot 0.05 \text{ [rad]}}} \tag{23.179}$$

Solution to Exercise 12.2

1. There a difference between the frequence responses of $H_{cont}(s)$ and $H_{disc}(z)$ because the discrete-time filter is derived from the continuous-time filter using an *approximation* of the time-derivatives in the filter model when that model is written as a differential equation, cf. Section 8.3 in the text-book.

   An explanation of why the difference increases with increasing frequency is that the approximation of the time-derivative becomes less accurate if signals vary faster (i.e. have higher frequency).

2. The frequency response curves of $H_{disc}(z)$ are unique up to the Nyquist frequency:

$$\underline{\underline{f_N}} = \frac{1}{T_s} = \frac{1}{0.2} = \underline{\underline{5 \text{ Hz}}} \tag{23.180}$$

$$\underline{\underline{\omega_N}} = \frac{\pi}{T_s} = \frac{\pi}{0.2} = \underline{\underline{5\pi \text{ rad/s} = 15.7 \text{ rad/s}}} \tag{23.181}$$

   which *is* the frequency indicated with a vertical line in Figure 12.1.

Solution to Exercise 13.1

The impulse response is equal to the transfer function:

$$y_\delta(z) = H(z) = \frac{1}{z - 0.5} \tag{23.182}$$

$$y_\delta(z) = H(z) \tag{23.183}$$

From Eq. (10.9) in the text-book,

$$y_\delta(k) = 0.5^k \tag{23.184}$$

The first four values of the impulse response is

$$\{0.5^0, 0.5^1, 0.5^2, 0.5^3, 0.5^4\} = \underline{\underline{\{1.0, 0.5, 0.25, 0.125, 0.0625\}}} \tag{23.185}$$

Obviously, the impulse response converges towards zero as time goes to infinity. Hence, the transfer function is <u>asymptotically stable</u>.

Solution of Exercise 13.2

The transfer function

$$H_1(s) = \frac{1}{z - 0.5} \tag{23.186}$$

is <u>asymptotically stable</u> since the pole $p = 0.5$ is inside the unit circle.

The transfer function

$$H_2(s) = \frac{1}{z + 0.5} \tag{23.187}$$

is <u>asymptotically stable</u> since the pole $p = -0.5$ is inside the unit circle.

The transfer function

$$H_3(s) = \frac{z - 2}{z - 0.5} \tag{23.188}$$

is <u>asymptotically stable</u> since the pole $p = 0.5$ is inside the unit circle. (The zero is 2, but the stability property is independent of the value of the zero.)

The transfer function

$$H_4(s) = \frac{1}{z - 1} \tag{23.189}$$

is <u>marginally stable</u> since the pole $p = 1$ is on the unit circle, and that pole is single.

The transfer function

$$H_5(s) = \frac{1}{z - 2} \tag{23.190}$$

is <u>unstable</u> since the pole $p = 2$ is outside the unit circle.

The transfer function

$$H_6(s) = \frac{1}{(z-1)^2} \tag{23.191}$$

is <u>unstable</u> since there are multiple (two) poles, namely $p_{1,2} = 1$, on the unit circle.

The transfer function

$$H_7(s) = \frac{1}{z} \tag{23.192}$$

is <u>asymptotically stable</u> since the pole $p = 0$ is inside the unit circle.

The transfer function

$$H_8(z) = \frac{1}{z^2 - 2.5z + 1} \tag{23.193}$$

is <u>unstable</u> since one of the poles is outside the unit circle. The poles are $p_{1,2} = 0.5,\ 2.0$.

Solution to Exercise 13.3

The pole of (13.11) is

$$p = 1 - \frac{T_s}{T} \tag{23.194}$$

The system is asymptotically stable if

$$\left| p = 1 - \frac{T_s}{T} \right| < 1 \tag{23.195}$$

If

$$1 - \frac{T_s}{T} > 0 \tag{23.196}$$

(23.195) becomes

$$1 - \frac{T_s}{T} < 1 \tag{23.197}$$

which gives

$$\underline{T_s > 0} \tag{23.198}$$

which is always satisfied.

If

$$1 - \frac{T_s}{T} < 0 \tag{23.199}$$

(23.195) becomes

$$-\left(1-\frac{T_s}{T}\right)<1 \tag{23.200}$$

which gives

$$\underline{T_s<\frac{T}{2}} \tag{23.201}$$

So, the system is asymptotically stable if

$$\underline{\underline{T_s<\frac{T}{2}}} \tag{23.202}$$

Solution to Exercise 13.4

The stability property is determined by the system eigenvalues, which are the roots of the characteristic equation:

$$\det(zI-A) = \det\left(z\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}1 & 0.5\\ 0 & 0.9\end{bmatrix}\right) \tag{23.203}$$

$$= \det\left(\begin{bmatrix}z-1 & 0.5\\ 0 & z-0.9\end{bmatrix}\right) \tag{23.204}$$

$$= (z-1)(z-0.9)-0\cdot 0.5 \tag{23.205}$$

$$= (z-1)(z-0.9) \tag{23.206}$$

The roots are

$$z_{1,2}=1,\ 0.9 \tag{23.207}$$

One pole is on the unit circle, and the other pole is inside the unit circle. Thererfore, the system is <u>marginally stable</u>.

Solution to Exercise 14.1

The loop transfer function is

$$L(z)=H_c(z)H_{pm}(z)=K_p\frac{z^{-1}}{1-z^{-1}} \tag{23.208}$$

The tracking function is

$$T(z)=\frac{L(z)}{1+L(z)}=\frac{K_p\frac{z^{-1}}{1-z^{-1}}}{1+K_p\frac{z^{-1}}{1-z^{-1}}}=\frac{K_p}{z-1+K_p}=\frac{K_p}{z-(1-K_p)} \tag{23.209}$$

The pole of $T(z)$ is

$$p=1-K_p \tag{23.210}$$

The control system is asymptotically stable if the absolute value of the pole is less than one:

$$|p|=|1-K_p|<1 \tag{23.211}$$

which is obtained with

$$\underline{\underline{0 < K_p < 2}} \tag{23.212}$$

Solution to Exercise 14.2

From the Bode diagram:

$$\underline{\underline{GM = 7 \text{ dB} = 2.2}} \tag{23.213}$$

$$\underline{\underline{PM = 35^\circ}} \tag{23.214}$$

$$\underline{\underline{\omega_c = 0.34 \text{ rad/s}}} \tag{23.215}$$

$$\underline{\underline{\omega_{180} = 0.58 \text{ rad/s}}} \tag{23.216}$$

Solution to Exercise 15.1

$$\{x(k)\} = \{x(0), x(1), x(2)\} = \{0.73, 1.23, 0.89\} \tag{23.217}$$

1. Mean value:

$$\underline{\underline{m_x}} = \frac{1}{N}\sum_{k=0}^{N-1} x(k) \tag{23.218}$$

$$= \frac{1}{3}\left[x(0) + x(1) + x(2)\right] \tag{23.219}$$

$$= \frac{1}{3}\left[0.73 + 1.23 + 0.89\right] \tag{23.220}$$

$$= \underline{\underline{0.95}} \tag{23.221}$$

2. Variance:

$$\underline{\underline{\sigma_x^2}} = \frac{1}{N-1}\sum_{k=0}^{N-1}\left[x(k) - m_x\right]^2 \tag{23.222}$$

$$\frac{1}{3-1}\sum_{k=0}^{3-1}\left[x(k) - m_x\right]^2 \tag{23.223}$$

$$= \frac{1}{2}\left\{\begin{array}{l} \left[x(0) - m_x\right]^2 \\ +\left[x(1) - m_x\right]^2 \\ +\left[x(2) - m_x\right]^2 \end{array}\right\} \tag{23.224}$$

$$= \frac{1}{2}\left\{\begin{array}{l} \left[0.73 - 0.95\right]^2 \\ +\left[1.23 - 0.95\right]^2 \\ +\left[1.13 - 0.95\right]^2 \end{array}\right\} \tag{23.225}$$

$$= \underline{\underline{0.0652}} \tag{23.226}$$

3. Standard deviation:

$$\underline{\underline{\sigma_x}} = \sqrt{\sigma_x^2} = \sqrt{0.0652} = \underline{\underline{0.255}} \tag{23.227}$$

4. Calculate the auto-covariance, $R_x(L)$ with $L = 0$ and $L = 1$ for the following options (the same options that are defined for cross-covariance in the text-book):

   (a) Raw estimate of auto-covariance: General formula:

$$R_x(L) = \sum_{k=0}^{N-1-|L|} [x(k+L)-m_x][x(k)-m_x] = R_x(L)_{\text{Raw}} \tag{23.228}$$

$L = 0$:

$$\begin{aligned}
\underline{\underline{R_x(0)}} &= \sum_{k=0}^{3-1-|0|=2} [x(k+0) - m_x][x(k) - m_x] && (23.229)\\
&= \sum_{k=0}^{2} [x(k) - m_x]^2 && (23.230)\\
&= [x(0) - m_x]^2 && (23.231)\\
&\quad +[x(1) - m_x]^2 && (23.232)\\
&\quad +[x(2) - m_x]^2 && (23.233)\\
&= [0.73 - 0.95]^2 && (23.234)\\
&\quad +[1.23 - 0.95]^2 && (23.235)\\
&\quad +[0.89 - 0.95]^2 && (23.236)\\
&= \underline{\underline{0.1304}} && (23.237)\\
&= R_x(0)_{\text{Raw}} && (23.238)
\end{aligned}$$

$L = 1$:

$$\begin{aligned}
\underline{\underline{R_x(1)}} &= \sum_{k=0}^{3-1-|1|=1} [x(k+1) - m_x][x(k) - m_x] && (23.239)\\
&= [x(1) - m_x][x(0) - m_x] && (23.240)\\
&\quad +[x(2) - m_x][x(1) - m_x] && (23.241)\\
&= [1.23 - 0.95][0.73 - 0.95] && (23.242)\\
&\quad +[0.89 - 0.95][1.23 - 0.95] && (23.243)\\
&= \underline{\underline{-0.0784}} && (23.244)\\
&= R_x(1)_{\text{Raw}} && (23.245)
\end{aligned}$$

(b) Normalized estimate: General formula:

$$\begin{aligned} R_x(L) &= \frac{1}{R_x(0)_{\text{Raw}}} \sum_{k=0}^{N-1-|L|} [x(k+L) - m_x][x(k) - m_x] \\ &= \frac{1}{R_x(0)_{\text{Raw}}} R_x(L)_{\text{Raw}} \end{aligned} \tag{23.246}$$

$L = 0$:

$$\underline{\underline{R_x(0)}} = \frac{1}{R_x(0)_{\text{Raw}}} R_x(0)_{\text{Raw}} = \underline{\underline{1}} \tag{23.247}$$

$L = 1$:

$$\underline{\underline{R_x(1)}} = \frac{1}{R_x(0)_{\text{Raw}}} R_x(1)_{\text{Raw}} = \frac{1}{0.1304}(-0.0784) = \underline{\underline{-0.6012}} \tag{23.248}$$

(c) Unbiased estimate: General formula:

$$R_x(L) = \frac{1}{N-L} R_x(L)_{\text{Raw}} \tag{23.249}$$

$L = 0$:

$$\underline{\underline{R_x(0)}} = \frac{1}{3-0} R_x(0)_{\text{Raw}} = \frac{1}{3} \cdot 0.1304 = \underline{\underline{0.0435}} \tag{23.250}$$

$L = 1$:

$$\underline{\underline{R_x(1)}} = \frac{1}{3-1} R_x(1)_{\text{Raw}} = \frac{1}{2}(-0.0784) = \underline{\underline{-0.0392}} \tag{23.251}$$

(d) Biased estimate: General formula:

$$R_x(L) = \frac{1}{N} R_x(L)_{\text{Raw}} \tag{23.252}$$

$L = 0$:

$$\underline{\underline{R_x(0)}} = \frac{1}{3} R_x(0)_{\text{Raw}} = \frac{1}{3} \cdot 0.1304 = \underline{\underline{0.0435}} \tag{23.253}$$

$L = 1$:

$$\underline{\underline{R_x(1)}} = \frac{1}{3} R_x(1)_{\text{Raw}} = \frac{1}{3}(-0.0784) = \underline{\underline{-0.0261}} \tag{23.254}$$

Solution to Exercise 15.2

1. Mean value:

$$\underline{\underline{m_x}} = \int_{-A}^{A} xP(x)dx \tag{23.255}$$
$$= \int_{-A}^{A} x \cdot \frac{1}{2A} \cdot dx \tag{23.256}$$
$$= \frac{1}{2A}\left[\frac{x^2}{2}\right]_{-A}^{A} \tag{23.257}$$
$$= \frac{1}{2A}\left[\frac{A^2}{2} - \frac{(-A)^2}{2}\right] \tag{23.258}$$
$$= \underline{\underline{0}} \tag{23.259}$$

2. Variance:

$$\underline{\underline{\sigma_x^2}} = \int_{-A}^{A} (x - m_x)^2 P(x)dx \tag{23.260}$$
$$= \int_{-A}^{A} (x - 0)^2 \frac{1}{2A} dx \tag{23.261}$$
$$= \frac{1}{2A}\int_{-A}^{A} x^2 dx \tag{23.262}$$
$$= \frac{1}{2A}\left[\frac{x^3}{3}\right]_{-A}^{A} \tag{23.263}$$
$$= \frac{1}{2A}\left[\frac{A^3}{3} - \frac{(-A)^3}{3}\right] \tag{23.264}$$
$$= \underline{\underline{\frac{A^2}{3}}} \tag{23.265}$$

Solution to Exercise 15.3

See Figure 23.11.

The standard deviation is

$$\underline{\underline{\sigma_x}} = \sqrt{\sigma_x^2} = \sqrt{4} = \underline{\underline{2}} \tag{23.266}$$

Solution to Exercise 15.4

1. With $a = 0$ the filter model is

$$x(k) = v(k) \tag{23.267}$$

So, if the input is white, the output is white.

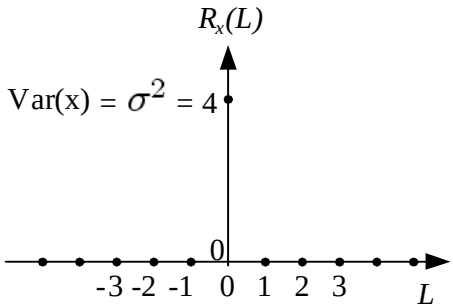

Figure 23.11:

2. If a is between 0 and 1, the filter output $x(k)$ depends not only on the input $v(k)$ but also on $x(k-1)$ which is x at the previous time step. Therefore, $x(k)$ will not vary purely randomly (it will not become purely white) – it is "coloured".

Solution to Exercise 15.5

y expressed as a function of u:

$$\underline{\underline{y = Gu + C}} \tag{23.268}$$

where G and C are calculated from the following formulas:

$$m_y = Gm_u + C \tag{23.269}$$

and

$$\sigma_y^2 = G^2\sigma_u^2 \tag{23.270}$$

where

$$m_u = 0 \tag{23.271}$$

$$m_y = 3 \tag{23.272}$$

$$\sigma_u^2 = 1 \tag{23.273}$$

$$\sigma_y^2 = 4 \tag{23.274}$$

Now, from (23.270) we get

$$\underline{\underline{G}} = \sqrt{\frac{\sigma_y^2}{\sigma_u^2}} = \sqrt{\frac{4}{1}} = \underline{\underline{2}} \tag{23.275}$$

and from (23.269) we get

$$\underline{\underline{C}} = m_y - Gm_u = 3 - 2 \cdot 0 = \underline{\underline{3}} \tag{23.276}$$

Solution to Exercise 16.1

Figure 23.12 shows the points, the line and the predictions errors, e_i.

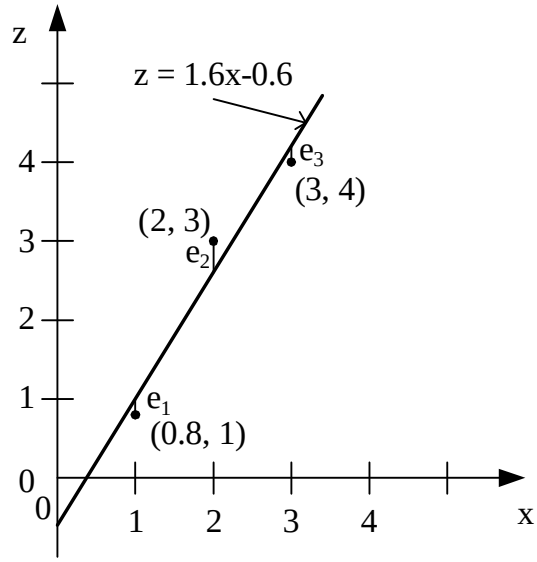

Figure 23.12:

Solution to Exercise 16.2

Writing the model on standard regression form:

$$h(k) - h(k-1) = \begin{bmatrix} -\sqrt{h(k-1)} & u(k-1) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{23.277}$$

The total model becomes

$$\underbrace{\begin{bmatrix} h(1)-h(0) \\ h(2)-h(1) \\ h(3)-h(2) \\ h(4)-h(3) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} -\sqrt{h(0)} & u(0) \\ -\sqrt{h(1)} & u(1) \\ -\sqrt{h(2)} & u(2) \\ -\sqrt{h(3)} & u(3) \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\theta} \tag{23.278}$$

Solution to Exercise 16.3

K is given by

$$
\begin{aligned}
y_k &= K && (23.279)\\
&= 1 \cdot K && (23.280)\\
&= \varphi\theta && (23.281)
\end{aligned}
$$

The LS estimate is

$$
\begin{aligned}
\theta_{\mathrm{LS}} &= K_{\mathrm{LS}} && (23.282)\\
&= (\Phi^T\Phi)^{-1}\Phi^T y && (23.283)\\
&= \left(\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} K_0 + e_1 \\ K_0 + e_2 \\ \vdots \\ K_0 + e_N \end{bmatrix} && (23.284)\\
&= N^{-1} \cdot \left[N \cdot K_0 + \sum_{k=1}^{N} e_k \right] && (23.285)\\
&= K_0 + \frac{1}{N} \sum_{k=1}^{N} e_k && (23.286)\\
&= K_0 + m_e && (23.287)
\end{aligned}
$$

where m_e is the mean value of the noise. From (23.287) you can see that K_{LS} *does not converge* towards K_0 if the mean value m_e is *different from zero.*

Solution to Exercise 16.4

According to the Parsimony Principle, you should select the smallest order with relatively small value of the criterion function. So, you should select order n_2.

Solution to Exercise 16.5

1.

$$\underline{\underline{V_{\text{norm}}}} = \frac{1}{m}\sum_{k=1}^{N}[y_k - (c_{LS}x_k + d_{LS})]^2 \tag{23.288}$$

$$= \frac{1}{3}\left\{\begin{array}{c}[y_1 - (c_{LS}\cdot x_1 + d_{LS})]^2 \\ +[y_2 - (c_{LS}\cdot x_2 + d_{LS})]^2 \\ +[y_3 - (c_{LS}\cdot x_3 + d_{LS})]^2\end{array}\right\} \tag{23.289}$$

$$= \frac{1}{3}\left\{\begin{array}{c}[0.8 - (1.6\cdot 1 - 0.6)]^2 \\ +[3.0 - (1.6\cdot 2 - 0.6)]^2 \\ +[4.0 - (1.6\cdot 3 - 0.6)]^2\end{array}\right\} \tag{23.290}$$

$$= \underline{\underline{0.08}} \tag{23.291}$$

2. With the two datasets

$$(y_{1,}x_1) = (0.8, 1.0) \tag{23.292}$$

$$(y_{2,}x_2) = (3.0, 2.0) \tag{23.293}$$

the model equations are

$$0.8 = c\cdot 1 + d = 2.2\cdot 1 + (-1.4) = 0.8 \tag{23.294}$$

$$3.0 = c\cdot 2 + d = 2.2\cdot 2 + (-1.4) = 3.0 \tag{23.295}$$

Hence, it is shown that $c = 2.2$ and $d = -1.4$ is a perfect match to these two data sets.

With $c = 2.2$ and $d = -1.4$ the criterion becomes

$$\underline{\underline{V_{\text{norm}}}} = \frac{1}{m}\sum_{k=1}^{N}[y_k - (cx_k + d)]^2 \tag{23.296}$$

$$= \frac{1}{3}\left\{\begin{array}{c}[y_1 - (c_{\text{LS}}\cdot x_1 + d_{\text{LS}})]^2 \\ +[y_2 - (c_{\text{LS}}\cdot x_2 + d_{\text{LS}})]^2 \\ +[y_3 - (c_{\text{LS}}\cdot x_3 + d_{\text{LS}})]^2\end{array}\right\} \tag{23.297}$$

$$= \frac{1}{3}\left\{\begin{array}{c}[0.8 - (2.2\cdot 1 - 1.4)]^2 \\ +[3.0 - (2.2\cdot 2 - 1.4)]^2 \\ +[4.0 - (2.2\cdot 3 - 1.4)]^2\end{array}\right\} \tag{23.298}$$

$$= \underline{\underline{0.48}} \tag{23.299}$$

which is larger than the LS-value $V_{\text{norm}} = 0.08$. Hence, it is demonstrated that $c = 2.2$ and $d = -1.4$ are not LS parameters!

Solution to Exercise 16.6

The order should be large enough to include the time-delay. One time step of 0.5 sec corresponds to a time-delay of 0.5 s, and one such time-delay is represented by the factor z^{-1} in the transfer function. The model should therefore include $2/0.5 = 4$ such factors. Hence, the mimimum order of the transfer function is $\underline{\underline{4}}$.

Solution to Exercise 16.7

The model written on the standard LS form:

$$\underbrace{\frac{K_T}{R_a}\left[v_a(t_k) - K_e\omega(t_k)\right]}_{y} = \underbrace{\begin{bmatrix} \dot{\omega}(t_k) & 1 \end{bmatrix}}_{\varphi}\underbrace{\begin{bmatrix} J \\ T_L \end{bmatrix}}_{\theta} \tag{23.300}$$

$\dot{\omega}(t_k)$ is calculated with the center difference method:

$$\dot{\omega}(t_k) \approx \frac{\omega(t_{k+1}) - \omega(t_{k-1})}{2T_s} \tag{23.301}$$

where T_s is the time step.

Solution to Exercise 16.8

You can remove the mean values of both the input and the output signals (time series or sequences), and use the devations as new input and output. In more detail: Assume that $\{u(k)\}$ is the original input signal and $\{y(k)\}$ is the output signal, and that m_u and m_y are the respective mean values. The deviation signals are then

$$\{du(k)\} = \{u(k) - m_u\} \tag{23.302}$$

and

$$\{dy(k)\} = \{y(k) - m_y\} \tag{23.303}$$

The signals $\{du(k)\}$ and $\{dy(k)\}$ are used as input and output signals.

Solution to Exercise 17.1

1. L is modeled as

$$\dot{L}(t) = 0 \tag{23.304}$$

(17.1) and (23.304) written as a state-space model (the time argument t is omitted for simplicity):

$$\dot{S} = \overbrace{\frac{1}{T_m}\left[-S + K_m\left(u + L\right)\right]}^{f_1} \tag{23.305}$$

$$\dot{L} = \underbrace{0}_{f_2} \tag{23.306}$$

The general observer formula for continuous-time implementation is

$$\dot{x}_e = f\left(x_e,\, u\right) + Ke \tag{23.307}$$

In detail this becomes

$$\underline{\underline{\dot{S}_e}} = f_1\left(S_e, L_3\right) + K_1 e = \underline{\underline{\frac{1}{T_m}\left[-S_e + K_m\left(u + L_e\right)\right] + K_1 e}} \tag{23.308}$$

$$\underline{\underline{\dot{L}_e}} = f_2\left(S_e, L_3\right) + K_2 e = \underline{\underline{K_2 e}} \tag{23.309}$$

where

$$e = S - S_e = \text{ Speed measurement } - \text{ Speed estimate} \tag{23.310}$$

2. The general observer formula for discrete-time implementation is

$$x_e(t_{k+1}) = x_e(t_k) + T_s\left[f(\cdot, t_k) + Ke(t_k)\right] \tag{23.311}$$

In detail this becomes

$$\underline{\underline{S_e(t_{k+1})}} = S_e\left(t_k\right) + T_s\left[f_1(\cdot, t_k) + K_1 e\left(t_k\right)\right] \tag{23.312}$$

$$= \underline{\underline{S_e\left(t_k\right) + T_s\left(\frac{1}{T_m}\left\{-S_e\left(t_k\right) + K_m\left[u\left(t_k\right) + L_e\left(t_k\right)\right]\right\} + K_1 e\right)}} \tag{23.313}$$

$$\underline{\underline{L_e(t_{k+1})}} = L_e\left(t_k\right) + T_s\left[f_2(\cdot, t_k) + K_2 e\left(t_k\right)\right] \tag{23.314}$$

$$= \underline{\underline{L_e\left(t_k\right) + T_s K_2 e\left(t_k\right)}} \tag{23.315}$$

3. To find the observer gains K_1 and K_2 we need a linearized state-space model on the form

$$\Delta\dot{x} = A\Delta x + B\Delta u \tag{23.316}$$

$$\Delta y = C\Delta x + D\Delta u \tag{23.317}$$

Linearization of the state-space model (23.305), (23.306) gives[1]

$$\Delta\dot{S} = \frac{\partial f_1}{\partial S}\Delta S + \frac{\partial f_1}{\partial L}\Delta L + \frac{\partial f_1}{\partial u}\Delta u \tag{23.318}$$

$$= -\frac{1}{T_m}\Delta S + \frac{K_m}{T_m}\Delta L + \frac{K_m}{T_m}\Delta u \tag{23.319}$$

[1] Since the original model is linear, it is actually not necessary to perform the linearization, but I do it here it anyway to demonstrate the procedure.

$$\Delta\dot{L} = \frac{\partial f_2}{\partial S}\Delta S + \frac{\partial f_2}{\partial L}\Delta L + \frac{\partial f_2}{\partial u}\Delta u \tag{23.320}$$
$$= 0 \cdot \Delta S + 0 \cdot \Delta L + 0 \cdot \Delta u \tag{23.321}$$
$$= 0 \tag{23.322}$$

The measurement is S. On matrix-vector form the linearized state-space model is

$$\begin{bmatrix} \Delta\dot{S} \\ \Delta\dot{L} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial L} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial L} \end{bmatrix} \begin{bmatrix} \Delta S \\ \Delta L \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} \Delta u \tag{23.323}$$
$$= \underbrace{\begin{bmatrix} -\frac{1}{T_m} & \frac{K_m}{T_m} \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \Delta S \\ \Delta L \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{K_m}{T_m} \\ 0 \end{bmatrix}}_{B} \Delta u \tag{23.324}$$

$$\Delta S = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} \Delta S \\ \Delta L \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} \Delta u \tag{23.325}$$

The eigenvalues of the observer error dynamics are the roots of the characteristic equation:

$$0 = \det\left[sI - (A - KC)\right] \tag{23.326}$$
$$= \det\left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} -\frac{1}{T_m} & \frac{K_m}{T_m} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \right\} \tag{23.327}$$
$$= \det\left\{ \begin{bmatrix} s + \frac{1}{T_m} + K_1 & -\frac{K_m}{T_m} \\ K_2 & s \end{bmatrix} \right\} \tag{23.328}$$
$$= s^2 + \left(\frac{1}{T_m} + K_1 \right) s + \frac{K_2 K_m}{T_m} \tag{23.329}$$

The second order polynomial (23.329) is to be compared with the second order Butterworth polynomial

$$B_2(s) = (Ts)^2 + 1.4142\,(Ts) + 1 = T^2 s^2 + 1.4142 Ts + 1 \tag{23.330}$$

where T is

$$T = \frac{T_r}{n} \tag{23.331}$$

where T_r is the specified observer response time and $n = 2$ is the order of the observer. Before comparing polynomials we divide (23.330) by T^2 so that it gets the same form as (23.329):

$$B_2^*(s) = s^2 + \frac{1.4142}{T} s + \frac{1}{T^2} \equiv s^2 + \left(\frac{1}{T_m} + K_1 \right) s + \frac{K_2 K_m}{T_m} \tag{23.332}$$

Comparing coefficients:

$$\frac{1.4142}{T} = \frac{1}{T_m} + K_1 \tag{23.333}$$

$$\frac{1}{T^2} = \frac{K_2 K_m}{T_m} \tag{23.334}$$

which gives

$$\underline{\underline{K_1 = \frac{1.4142}{T} - \frac{1}{T_m} = \frac{1.4142n}{T_r} - \frac{1}{T_m}}} \tag{23.335}$$

$$\underline{\underline{K_2 = \frac{T_m}{K_m T^2} = \frac{T_m n^2}{K_m T_r^2}}} \tag{23.336}$$

4. The observability test is made with the matrices A and C of the linear state space model. The observability matrix is ($n = 2$)

$$M_{\text{obs}} = \begin{bmatrix} C \\ CA^{2-1} = CA \end{bmatrix} \tag{23.337}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ - - - - - - - - - - - - \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{T_m} & \frac{K_m}{T_m} \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{23.338}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{T_m} & \frac{K_m}{T_m} \end{bmatrix} \tag{23.339}$$

The determinant of M_{obs} is

$$\det(M_{\text{obs}}) = 1 \cdot \left(\frac{K_m}{T_m}\right) - \left(-\frac{1}{T_m}\right) \cdot 0 = \frac{K_m}{T_m} \tag{23.340}$$

Since $\frac{K_m}{T_m} \neq 0$, the rank of M_{obs} is full (2), and the system is <u>observable</u>.

5. To speed up the response of the estimates, <u>the observer response time T_r can be decreased</u>. This will increase the observer gains, and the estimates become <u>more noisy</u>.

6. To make the estimated become smoother you can <u>let the estimates pass through lowpass filters</u>.

7. The feedforward control function is derived from the motor model, which is

$$T_m \dot{S}(t) + S(t) = K_m[u(t) + L(t)] \tag{23.341}$$

In this model, the speed is substituted by its reference or setpoint S_r, the load is substituted by its estimate, and then the equation is solved for the control variable. The result is

$$\underline{\underline{u_f(t) = \frac{1}{K_m} T_m \dot{S}_r(t) + S_r(t) - L_e(t)}} \tag{23.342}$$

In Exercise 18.1 experimental results with feedforward control with this motor are shown. In that exercise a Kalman Filter is used in stead of an observer, but you can expect the same results with an observer.

8. Increased robustness against sensor failure can be obtained as follows: The feedback to the speed controller (e.g. a PI controller) is permanently based on the estimated measurement, S_e, as calculated by an observer. If the sensor is failing (assuming some kind of measurement error detection has been implemented, of course), the estimates are *not updated* by the (erroneous) measurement. This can be implemented by multiplying $K_1 e$ in (23.308) and $K_2 e$ in (23.309) by zero so that the effective continuous-time observer formulas are

$$\dot{S}_e = \frac{1}{T_m} \left[-S_e + K_m \left(u + L_e \right) \right] \tag{23.343}$$

$$\dot{L}_e = 0 \tag{23.344}$$

The discrete-time observer formulas are

$$S_e(t_{k+1}) = S_e\left(t_k\right) + T_s \left(\frac{1}{T_m} \left\{ -S_e\left(t_k\right) + K_m \left[u\left(t_k\right) + L_e\left(t_k\right) \right] \right\} \right) \tag{23.345}$$

$$L_e(t_{k+1}) = L_e\left(t_k\right) \tag{23.346}$$

Hence, the observer is just a *simulator* of the motor. So, during sensor failure, the speed controller acts on basis of the simulated speed, and this is better than acting on basis of an erroneous speed measurement.

In Exercise 18.1 experimental results with sensor failure with this motor are shown. In that exercise a Kalman Filter is used in stead of an observer, but you can expect the same results with an observer.

Solution to Exercise 17.2

1. According to Figure 17.1 in the text-book, the updating of the time-derivative of the estimates are *proportional* to the measurement estimation error:
$$\dot{x}_e = f\left(x_e,\, u, w_k\right) + Ke \tag{23.347}$$
so it is proportional compensation.

2. In control theory it is well-known that proportional + integral control makes the steady-state control error become zero. So, it is reasonable to suggest that a *proportional + integral* (PI) based compensator in an observer can may remove the steady-state estimation error. Actually, it can be shown that such a proportional + integral compensation is obtained with augmenting the original state variables with state variables representing disturbances to be estimated, cf. Section 17.6 in the text-book. And such state augmentation may remove the steady-state estimation error.

Solution to Exercise 17.3

1. State equations:
$$\dot{p} = \underbrace{v}_{f_1} \tag{23.348}$$
$$\dot{v} = \underbrace{a}_{f_2} \tag{23.349}$$
$$\dot{a} = \underbrace{0}_{f_3} \tag{23.350}$$
Measurement:
$$y = p \tag{23.351}$$

2. The general observer formula for continuous-time implementation is
$$\dot{x}_e = f\left(x_e,\, u\right) + Ke \tag{23.352}$$
In detail this becomes
$$\underline{\underline{\dot{p}_e}} = f_1\left(p_e, v_e, a_e\right) + K_1 e = \underline{\underline{v_e + K_1 e}} \tag{23.353}$$
$$\underline{\underline{\dot{v}_e}} = f_2\left(p_e, v_e, a_e\right) + K_2 e = \underline{\underline{a_e + K_2 e}} \tag{23.354}$$
$$\underline{\underline{\dot{a}_e}} = f_3\left(p_e, v_e, a_e\right) + K_3 e = \underline{\underline{K_3 e}} \tag{23.355}$$
where
$$e = p - p_e = \text{ Position measurement } - \text{ Position estimate} \tag{23.356}$$

3. The general observer formula for discrete-time implementation is

$$x_e(t_{k+1}) = x_e(t_k) + T_s\left[f(\cdot, t_k) + Ke(t_k)\right] \tag{23.357}$$

In detail this becomes

$$\underline{\underline{p_e(t_{k+1})}} = p_e\left(t_k\right) + T_s\left[f_1(\cdot, t_k) + K_1 e\left(t_k\right)\right] = \underline{\underline{p_e\left(t_k\right) + T_s\left[v_e(t_k) + K_1 e\left(t_k\right)\right]}} \tag{23.358}$$

$$\underline{\underline{v_e(t_{k+1})}} = v_e\left(t_k\right) + T_s\left[f_2(\cdot, t_k) + K_2 e\left(t_k\right)\right] = \underline{\underline{v_e\left(t_k\right) + T_s\left[a_e(t_k) + K_2 e\left(t_k\right)\right]}} \tag{23.359}$$

$$\underline{\underline{a_e(t_{k+1})}} = a_e\left(t_k\right) + T_s\left[f_3(\cdot, t_k) + K_3 e\left(t_k\right)\right] = \underline{\underline{a_e\left(t_k\right) + T_s\left[K_3 e\left(t_k\right)\right]}} \tag{23.360}$$

4. To find the observer gains K_1 and K_2 we need a linearized state-space model on the form

$$\Delta\dot{x} = A\Delta x + B\Delta u \tag{23.361}$$

$$\Delta y = C\Delta x + D\Delta u \tag{23.362}$$

Linearization of the state-space model (23.348) – (23.350) gives[2]

$$\Delta\dot{p} = \frac{\partial f_1}{\partial p}\Delta p + \frac{\partial f_1}{\partial v}\Delta v + \frac{\partial f_1}{\partial a}\Delta a = 0\cdot\Delta p + 1\cdot\Delta v + 0\cdot\Delta a \tag{23.363}$$

$$\Delta\dot{v} = \frac{\partial f_2}{\partial p}\Delta p + \frac{\partial f_2}{\partial v}\Delta v + \frac{\partial f_2}{\partial a}\Delta a = 0\cdot\Delta p + 0\cdot\Delta v + 1\cdot\Delta a \tag{23.364}$$

$$\Delta\dot{a} = \frac{\partial f_3}{\partial p}\Delta p + \frac{\partial f_3}{\partial v}\Delta v + \frac{\partial f_3}{\partial a}\Delta a = 0\cdot\Delta p + 0\cdot\Delta v + 0\cdot\Delta a \tag{23.365}$$

The measurement is p. On matrix-vector form the linearized state-space model is (the model has no $B\Delta u$-term and no $D\Delta u$-term)

$$\begin{bmatrix} \Delta\dot{p} \\ \Delta\dot{v} \\ \Delta\dot{a} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial p} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial a} \\ \frac{\partial f_2}{\partial p} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial a} \\ \frac{\partial f_3}{\partial p} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial a} \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix} \tag{23.366}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix} \tag{23.367}$$

Measurement equation:

$$\Delta p = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix} \tag{23.368}$$

[2] Since the original model is linear, it is actually not necessary to perform the linearization, but I do it here anyway to demonstrate the procedure.

The eigenvalues of the observer error dynamics are the roots of the characteristic equation:

$$
\begin{aligned}
0 &= \det\left[sI-(A-KC)\right] && (23.369)\\
&= \det\left\{\begin{bmatrix} s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{bmatrix} - \left(\begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} - \begin{bmatrix} K_1\\ K_2\\ K_3\end{bmatrix}\begin{bmatrix} 1 & 0 & 0\end{bmatrix}\right)\right\} && (23.370)\\
&= \det\begin{bmatrix} s+K_1 & -1 & 0\\ K_2 & s & -1\\ K_3 & 0 & s\end{bmatrix} && (23.371)\\
&= (s+K_1)\det\begin{bmatrix} s & -1\\ 0 & s\end{bmatrix} - && (23.372)\\
&\quad (-1)\cdot\det\begin{bmatrix} K_2 & -1\\ K_3 & s\end{bmatrix} && (23.373)\\
&\quad +0\cdot\det\begin{bmatrix} K_2 & s\\ K_3 & 0\end{bmatrix} && (23.374)\\
&= (s+K_1)s^2-(-1)\cdot(K_2s+K_3)+0 && (23.375)\\
&= s^3+K_1s^2+K_2s+K_3 && (23.376)
\end{aligned}
$$

The third order polynomial (23.376) is to be compared with the third order Butterworth polynomial

$$B_3(s) = (Ts)^3 + 2\,(Ts)^2 + 2\,(Ts) + 1 \qquad (23.377)$$

where T is

$$T = \frac{T_r}{n} \qquad (23.378)$$

where T_r is the specified observer response time and $n = 3$ is the order of the observer. Before comparing polynomials we divide (23.377) by T^3 so that it gets the same form as (23.376):

$$B_3^*(s) = s^3 + \frac{2}{T}s^2 + \frac{2}{T^2}s + \frac{1}{T^3} \equiv s^3 + K_1s^2 + K_2s + K_3 \qquad (23.379)$$

Comparing coefficients gives

$$\underline{\underline{K_1 = \frac{2}{T} = \frac{2n}{T_r}}} \qquad (23.380)$$

$$\underline{\underline{K_2 = \frac{2}{T^2} = \frac{2n^2}{T_r^2}}} \qquad (23.381)$$

$$\underline{\underline{K_3 = \frac{1}{T^3} = \frac{n^3}{T_r^3}}} \qquad (23.382)$$

Solution to Exercise 18.1

1. L is modeled as

$$\dot{L}(t) = 0 \tag{23.383}$$

(18.1) and (23.383) written as a state-space model (the time argument t is omitted for simplicity):

$$\dot{S} = \overbrace{\frac{1}{T_m}\left[-S + K_m\left(u + L\right)\right]}^{f_{\text{cont}_1}} \tag{23.384}$$

$$\dot{L} = \underbrace{0}_{f_{\text{cont}_2}} \tag{23.385}$$

Applying Forward discretization with time step T_s and including white disturbance noise in the resulting difference equations yields the following discrete time state space model of the motor:

$$\begin{aligned} S(k+1) &= S(k) + T_s f_{\text{cont}_1}(\cdot, k) + w_1(k) && (23.386)\\ &= \underbrace{S(k) + \frac{T_s}{T_m}\left\{-S(k) + K_m\left[u(k) + L(k)\right]\right\}}_{f_1} && (23.387)\\ &\quad + w_1(k) && (23.388) \end{aligned}$$

$$\begin{aligned} L(k+1) &= L(k) + T_s f_{\text{cont}_2}(\cdot, k) + w_2(k) && (23.389)\\ &= \underbrace{L(k)}_{f_2} + w_2(k) && (23.390) \end{aligned}$$

w_1 and w_2 are independent (uncorrelated) white process noises with assumed variances Q_1 and Q_2, respectively. The noise vector becomes

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{23.391}$$

having covariance

$$Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} \tag{23.392}$$

The speed S is measured, so we add the following measurement equation to the state space model:

$$y(k) = S(k) + v(k) \tag{23.393}$$

where v is white measurement noise with assumed variance R.

The Kalman Filter formulas are as follows.

It is assumed that the predicted speed estimate $S_p(k)$ exists (initially it is a "guess" of the estimate, and later it is predicted using the model, see below).

The innovation variable is calculated as

$$e(k) = S(k) - S_p(k) \tag{23.394}$$

where $S(k)$ is the speed measurement.

The corrected state estimates – which are used as applied estimates – are

$$\begin{aligned} S_c(k) &= S_p(k) + K_{11}e(k) \\ L_c(k) &= L_p(k) + K_{21}e(k) \end{aligned} \tag{23.395}$$

The predicted state estimates for the next time step are

$$\begin{aligned} S_p(k+1) &= S_c(k) + \tfrac{T_s}{T_m}\left\{-S_c(k) + K_m\left[u(k) + L_c(k)\right]\right\} \\ L_p(k+1) &= L_c(k) \end{aligned} \tag{23.396}$$

2. The quantities needed for calculating the steady-state Kalman Filter gain, K, are indicated in Figure 23.13. These quantities are given below. Transition matrix of linearized model:

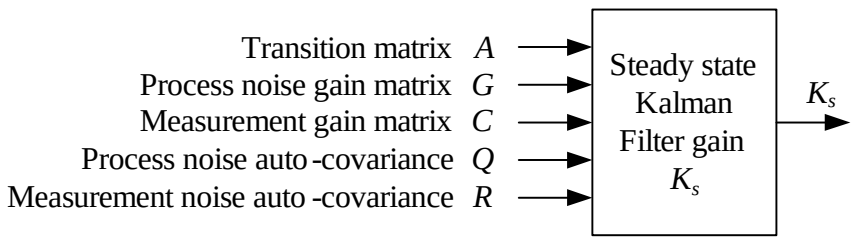

Figure 23.13: Illustration of what information is needed to compute the steady-state Kalman Filter gain, K_s.

$$A = I + T_s \left.\frac{\partial f_{\text{cont}}}{\partial x}\right|_{S_p(k), L_p(k), u(k)} \tag{23.397}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + T_s \left.\begin{bmatrix} \frac{\partial f_{\text{cont}_1}}{\partial x_1} = -\frac{1}{T_m} & \frac{\partial f_{\text{cont}_1}}{\partial x_2} = \frac{K_m}{T_m} \\ \frac{\partial f_{\text{cont}_2}}{\partial x_1} = 0 & \frac{\partial f_{\text{cont}_2}}{\partial x_2} = 0 \end{bmatrix}\right|_{S_p(k), L_p(k), u(k)} \tag{23.398}$$

$$= \begin{bmatrix} 1 - \frac{T_s}{T_m} & \frac{T_s K_m}{T_m} \\ 0 & 1 \end{bmatrix} \tag{23.399}$$

Process noise gain matrix:

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{23.400}$$

Measurement gain matrix:

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{23.401}$$

(since the measurement is $S = C \begin{bmatrix} S & L \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} S & L \end{bmatrix}^T$).

Process-noise covariance:

$$Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} \tag{23.402}$$

Measurement-noise covariance:

$$R \tag{23.403}$$

3. The observability test is made with the matrices A and C. The observability matrix is ($n = 2$)

$$M_{\text{obs}} = \begin{bmatrix} C \\ CA^{2-1} = CA \end{bmatrix} \tag{23.404}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 - \frac{T_s}{T_m} & \frac{T_s K_m}{T_m} \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{23.405}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{T_s}{T_m} & \frac{T_s K_m}{T_m} \end{bmatrix} \tag{23.406}$$

The determinant of M_{obs} is

$$\det(M_{\text{obs}}) = 1 \cdot \left(\frac{T_s K_m}{T_m}\right) - \left(-\frac{T_s}{T_m}\right) \cdot 0 = \frac{T_s K_m}{T_m} \tag{23.407}$$

Since $\frac{T_s K_m}{T_m} \neq 0$, the rank of M_{obs} is full (2), and the system is <u>observable</u>.

4. A reasonable value of the measurement noise covariance R can be found by
<u>calculating the variance of a sequence (time series) of the measurement</u>.

5. To speed up the response of the estimate $L_{est} = L_c$,
<u>the noise variance Q_{22} can be increased</u>. This will increase the corresponding Kalman Filter gain, and the estimate becomes <u>more noisy</u>, which is a drawback of course.

6. To make the estimates become smoother you can
   let the estimates pass through lowpass filters.

7. The feedforward control function is derived from the motor model, which is

$$T_m \dot{S}(t) + S(t) = K_m[u(t) + L(t)] \tag{23.408}$$

   In this model, the speed is substituted by its reference or setpoint S_r, the load is substituted by its estimate, and then the equation is solved for the control variable. The result is, in discrete time ($t_k \triangleq k$):

$$u_f(t_k) = \frac{1}{K_m} T_m \dot{S}_r(t_k) + S_r(t_k) - L_{est}(t_k) \tag{23.409}$$

   where $\dot{S}_r(t_k)$ can be calculated with e.g. Backward discretization:

$$\dot{S}_r(t_k) \approx \frac{S_r(t_k) - S_r(t_{k-1})}{T_s} \tag{23.410}$$

   Here are results of *experiments* made with the motor shown in Figure 18.1:

   - **Without feedforward**: Figure 23.17 (Page 150) shows the speed reference and measurement, the estimated load, and the control signal. The speed is controlled with a PI controller (with gain 0.5 and integral time 2 sec). The motor was braked with an approximately constant load torque (using the hand). The estimate gives a good estimate of the applied load torque - qualitatively.
   - **With feedforward**: Figure 23.18 (Page 151) shows responses with feedforward control (together with feedback control with PI controller, as in the previous experiment). It is obvious that the control system now performs much better since the control error is much smaller compared with not using feedforward.

8. Increased robustness against sensor failure can be obtained as follows: The feedback to the speed controller (e.g. a PI controller) is permanently based on the estimated measurement, S_e, as calculated by a Kalman Filter. If the sensor is failing (assuming some kind of measurement error detection has been implemented, of course), the (erroneous) measurement is removed from the corrected estimates (23.395). This can be implemented by multiplying Ke in (23.395) by zero. In this situation, the Kalman Filter is just a *simulator* of the motor. So, during sensor failure, the speed controller acts on basis of the simulated speed, and this is better than acting on basis of an erroneous speed measurement.

   Here are results of *experiments* made with the motor:

- **Full Kalman Filter - despite sensor failure**: Figure 23.19 (Page 152) shows the speed reference and measurement and the control signal. Sensor fails at time $t = 10$ sec, as the sensor signal suddenly becomes zero. This fail lasts for the whole experiment. The speed controller (a PI controller) "thinks" that the speed is actually zero, and therefore adjusts the control signal to its maximum in an attempt to increase the speed to become equal to the speed reference. This makes the motor to speed up maximum speed – much larger than the reference. No good...
- **Kalman Filter without measurement-based updating during sensor failure (i.e. Kalman Filter running as simulator)**: As in the previous case, the sensor fails at time $t = 10$ sec. Figure 23.20 (Page 153) shows the responses. Now the speed controller acts on basis of the simulated speed. This largely improves the behaviour of the control system. The control system is even able to track the speed reference change. So, it is demonstrated that it is much better to let the controller act on basis of a simulated measurement than an erroneous measurement.

Solution to Exercise 19.1

Figure 23.14 shows the control system that can be used in a simulator (in e.g. Simulink or LabVIEW). You can run the simulations with model parameters (indexed with 0 in the figure) in the simulated motor that are *different* from the model parameters (indexed with 1 in the figure) used in the feedback controller, the feedback controller and the estimator. Measurement noise is included in the simulator using a proper random signal generator (such signal generators exist in Simulink and LabVIEW).

Parameters K_{m_0} and T_{m_0} are the parameters you assume are accurately known and therefore use in the feedback controller, the feedback controller and the estimator. Parameters K_{m_1} and T_{m_1} are parameters you can use in the simulated motor when you want to introduce model errors. You can for example systematically vary K_{m_1} between

$$0.8K_{m_0} \leq K_{m_1} \leq 1.2K_{m_0} \tag{23.411}$$

and vary T_{m_1} between

$$0.8T_{m_0} \leq T_{m_1} \leq 1.2T_{m_0} \tag{23.412}$$

and run simulations with each of the different parameter sets.

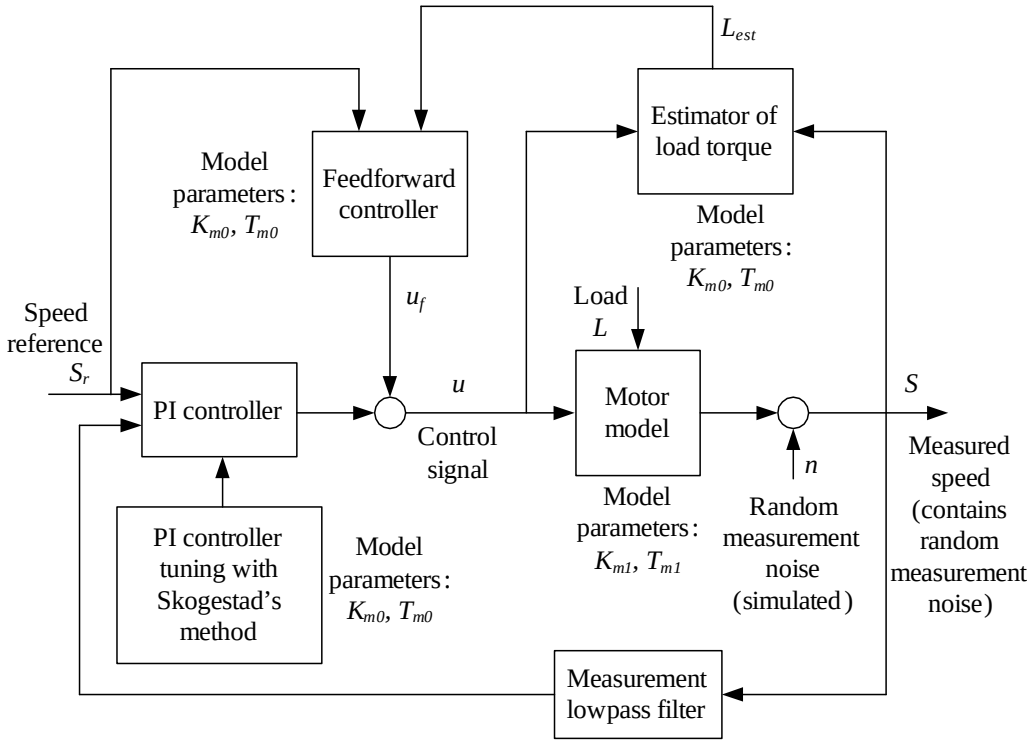

Figure 23.14:

Solution to Exercise 20.1

1. The standard process model form is

$$\dot{x} = f + Bu \tag{23.413}$$

We have

$$x = \begin{bmatrix} L \\ T \end{bmatrix} \tag{23.414}$$

and

$$u = \begin{bmatrix} F_c \\ F_h \end{bmatrix} \tag{23.415}$$

We get

$$\underline{\underline{B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho A} & \frac{1}{\rho A} \\ \frac{T_c - T}{\rho A L} & \frac{T_h - T}{\rho A L} \end{bmatrix}}} \tag{23.416}$$

and

$$\underline{\underline{f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -\frac{F}{\rho A} \\ 0 \end{bmatrix}}} \tag{23.417}$$

2. The control function is generally

$$u = B^{-1}\left(K_p e + K_i \int_0^t e d\tau + \dot{r}_f - f\right) \tag{23.418}$$

where (I use indices L and T instead of 1 and 2 as in the standard form)

$$u = \begin{bmatrix} F_c \\ F_h \end{bmatrix} \tag{23.419}$$

$$B = \begin{bmatrix} \frac{1}{\rho A} & \frac{1}{\rho A} \\ \frac{T_c - T}{\rho A L} & \frac{T_h - T}{\rho A L} \end{bmatrix} \tag{23.420}$$

$$e = \begin{bmatrix} L_r - L \\ T_r - T \end{bmatrix} \tag{23.421}$$

$$K_p = \begin{bmatrix} K_{p_L} & 0 \\ 0 & K_{p_T} \end{bmatrix} \tag{23.422}$$

$$K_i = \begin{bmatrix} K_{i_L} & 0 \\ 0 & K_{i_T} \end{bmatrix} = \begin{bmatrix} \frac{K_{p_L}}{T_{i_L}} & 0 \\ 0 & \frac{K_{p_T}}{T_{i_T}} \end{bmatrix} \tag{23.423}$$

$$\dot{r}_f = \begin{bmatrix} L_{r_f} \\ T_{r_f} \end{bmatrix} \tag{23.424}$$

$$f = \begin{bmatrix} -\frac{F}{\rho A} \\ 0 \end{bmatrix} \tag{23.425}$$

(where index f means lowpass filtered).

3.

$$\underline{\underline{K_{p_L} = \frac{1}{T_{C_L}}}} \tag{23.426}$$

$$\underline{\underline{T_{i_L} = kT_{C_L}}} \tag{23.427}$$

$$\underline{\underline{K_{p_T} = \frac{1}{T_{C_T}}}} \tag{23.428}$$

$$\underline{\underline{T_{i_T} = kT_{C_T}}} \tag{23.429}$$

where k can be set to 1.5.

4. The following parameters and variables must be known:

- ρ
- A
- L (using a level sensor)
- T (temperature sensor)
- T_c (temperature sensor)
- T_h (temperature sensor)
- F (flow sensor)

5. <u>No</u>

   <u>No</u>

6. <u>Yes</u>

Solution to Exercise 20.2

1. To derive the control function we first write the process model on the standard form:

$$\ddot{y} = \underbrace{\frac{1}{m}\left[-D|\dot{y}-u_c|\left(\dot{y}-u_c\right)+F_w(\phi,V_w)\right]}_{f} + \underbrace{\frac{1}{m}}_{B}\underbrace{F_t}_{u} \tag{23.430}$$

   The control function becomes

$$\underline{\underline{F_t}} = B^{-1}\left[K_p e + \frac{K_p}{T_i}\int_0^t e d\tau + K_p T_d \frac{de_f}{dt} + \ddot{r}_f - f\right] \tag{23.431}$$

$$= \underline{\underline{\left(\frac{1}{m}\right)^{-1}\left[\begin{array}{l} K_p e + \frac{K_p}{T_i}\int_0^t e d\tau + K_p T_d \frac{de_f}{dt} + \ddot{r}_f \\ \\ +\frac{1}{m}\left[-D|\dot{y}-u_c|\left(\dot{y}-u_c\right)+F_w(\phi,V_w)\right] \end{array}\right]}} \tag{23.432}$$

   where e is the control error:

$$e = r - y \tag{23.433}$$

   Index f in $\ddot{r}_f$ means that the reference is lowpass filtered (with a second order filter).

   The internal PID controller in (23.432) is on parallel form. Skogestad's PID tuning method for the double integrator (which is the linearized process for which the internal PID controller is

designed) assumes a serial PID controller. The PID settings for the serial PID controller becomes:

$$K_{p_s} = \frac{1}{4\left(T_C/2\right)^2} = \frac{1}{{T_C}^2} \tag{23.434}$$

$$T_{i_s} = 4T_C/2 = 2T_C \tag{23.435}$$

$$T_{d_s} = 4T_C/2 = 2T_C \tag{23.436}$$

Using the formulas for transformation of serial PID parameters to parallel PID parameters, we have

$$\underline{\underline{K_p = K_{p_p}}} = K_{p_s}\left(1 + \frac{T_{d_s}}{T_{i_s}}\right) = \frac{1}{{T_C}^2}\left(1 + \frac{2T_C}{2T_C}\right) = \underline{\underline{\frac{2}{{T_C}^2}}} \tag{23.437}$$

$$\underline{\underline{T_i = T_{i_p}}} = T_{i_s}\left(1 + \frac{T_{d_s}}{T_{i_s}}\right) = 2T_C\left(1 + \frac{2T_C}{2T_C}\right) = \underline{\underline{4T_C}} \tag{23.438}$$

$$\underline{\underline{T_s = T_{d_p}}} = T_{d_s}\frac{1}{1 + \frac{T_{d_s}}{T_{i_s}}} = 2T_C\frac{1}{1 + \frac{2T_C}{2T_C}} = \underline{\underline{T_C}} \tag{23.439}$$

2. Of course, all variables and parameters of the control function (23.432) must be known to make the controller implementable. In more details:

   - The parameters m and D must be known.
   - The ship position y must be measured usinig e.g. GPS measurement.
   - The ship speed $\dot{y}$ can be calculated as the time-derivative of y, or it can be estimated with a Kalman Filter or an observer.
   - The water speed u_c must either be measured or estimated with a Kalman Filter or an observer.[3]
   - The wind angle ϕ and the wind speed V_w must be measured with a wind sensor (mounted on the ship). The wind force function F_w is assumed to be known for a given ship.[4]

3. In (23.432) the control variable F_t is a function of the disturbance $F_w(\phi, V_w)$. This dependency implements feedforward. The controller will compensate for the disturbance, so that the ship position y will not be influenced by F_w, whatever value of F_w.

[3] Kongsberg Maritime (Norway) use a Kalman Filter in their ship positioning systems – or DP (Dynamic Positioning) systems.

[4] F_w is calculated from the geometry of the ship.

Solution to Exercise 21.1

1.

$$\underline{\underline{\dot{x}_1 = \overbrace{x_2}^{f_1}}} \tag{23.440}$$

$$\underline{\underline{\dot{x}_2 = \overbrace{\frac{1}{m}\left[-D|x_2 - u_c|\,(x_2 - u_c) + F_w(\phi, V_w) + F_t\right]}^{f_2}}} \tag{23.441}$$

2. See Figure 23.15.

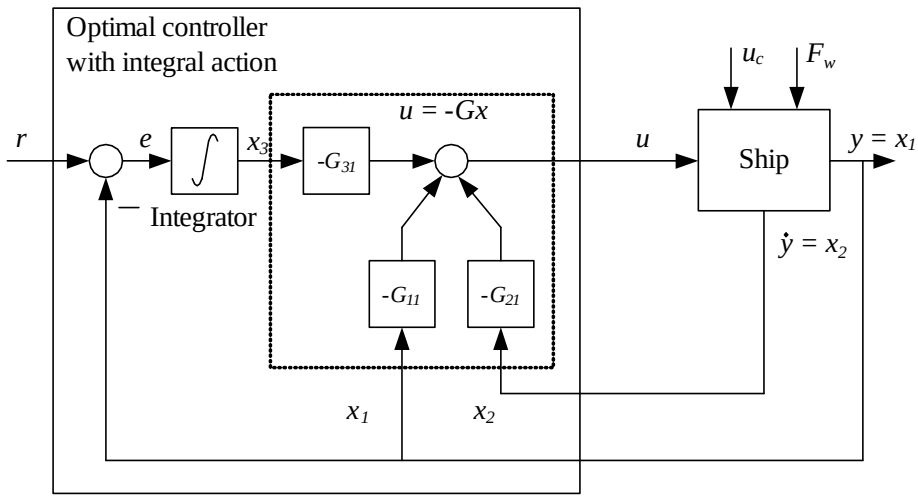

Figure 23.15:

3. See Figure 23.15. The state-variable of the integator is defined by

$$\dot{x}_3 = r - x_1 \tag{23.442}$$

The augmentet (total) state-space model becomes

$$\underline{\underline{\dot{x}_1 = \overbrace{x_2}^{f_1}}} \tag{23.443}$$

$$\underline{\underline{\dot{x}_2 = \overbrace{\frac{1}{m}\left[-D|x_2 - u_c|\,(x_2 - u_c) + F_w(\phi, V_w) + F_t\right]}^{f_2}}} \tag{23.444}$$

$$\underline{\underline{\dot{x}_3 = \overbrace{r - x_1}^{f_3}}} \tag{23.445}$$

4. Matrices A and B are found by linearization:

$$\underline{\underline{A}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{2D}{m}|x_2 - u_c| & 0 \\ 0 & 0 & -1 \end{bmatrix}}} \tag{23.446}$$

$$\underline{\underline{B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix}}} \tag{23.447}$$

(Element $A(2,2)$ can be found by resolving the absolute value by first assuming $(x_2 - u_c)$ is positive and then assuming $(x_2 - u_c)$ is negative, then taking the partial derivative, and finally expressing the results of the partial differentiations compactly.)

State weight matrix:

$$\underline{\underline{Q = \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{|x_{1_{\max}}|^2} & 0 & 0 \\ 0 & \frac{1}{|x_{2_{\max}}|^2} & 0 \\ 0 & 0 & \frac{1}{|x_{3_{\max}}|^2} \end{bmatrix}}} \tag{23.448}$$

State weight matrix:

$$\underline{\underline{R = \begin{bmatrix} R_{11} \end{bmatrix} = \begin{bmatrix} \frac{1}{|u_{\max}|^2} \end{bmatrix}}} \tag{23.449}$$

5. To reduce the fuel consumption (less aggressive control) you can increase the cost (weight) of the control signal, i.e. increase R_{11}.

6. To reduce the speed of the ship you can increase the cost (weight) of the speed, i.e. increase Q_{22}.

Solution to Exercise 22.1

<u>No.</u>

Solution to Exercise 22.2

1. The optimization criterion is

$$\begin{aligned} J &= \sum_{k=0}^{N_p} \left\{ Q_1 \left[e_1(t_k)\right]^2 + Q_2 \left[e_2(t_k)\right]^2 + ... + Q_n \left[e_n(t_k)\right]^2 \right\} \\ &\quad + \sum_{k=1}^{N_c} R_1 \left\{ \left[\Delta u_1(t_k)\right]^2 + R_2 \left[\Delta u_2(t_k)\right]^2 + ... + R_r \left[\Delta u_r(t_k)\right]^2 \right\} \end{aligned} \tag{23.450}$$

The state variables are $x_1 = y$ and $x_2 = \dot{y}$. Thererfore, the error e_1 is the position control error. To allow for a larger e_1 the cost or weight of e_1 – which is Q_1 – should be increased.

2. Yes, MPC takes into account specific limits of control variable(s) when calculating the optimal control signal.

Solution to Exercise 22.3

Cascade control.

r_{1MPC} may be level of a tank. r_{PID_1} may be flow reference (setpoint) to a flow PI(D) controller. u_1 may be control signal to a control valve.

Solution to Exercise 23.1

1. The dead-time is *larger* than the time-constant. Therefore you can expect improved disturbance compensation with dead-time compensation.

   Also, reference tracking will be improved with dead-time compensation, because dead-time compensation always improves reference tracking – no matter what the relation between the time-constant and the time-delay is.

2. The dead-time is *smaller* than the time-constant. Therefore you can not expect improved disturbance compensation with dead-time compensation.

   Reference tracking will be improved with dead-time compensation.

Solution to Exercise 23.2

According to Skogestad's method (with $c = 2$), the parameters of a PI controller is

$$K_p = \frac{T}{K\left(T_C + \tau\right)} = \frac{20}{2\left(10 + 0\right)} = 1 \tag{23.451}$$

$$T_i = \min\left[T,\, c\left(T_C + \tau\right)\right] = \min\left[20,\, 2\left(10 + 0\right)\right] = 20\ s \tag{23.452}$$

$$T_d = 0 \tag{23.453}$$

Solution to Exercise 23.3

1. The transportation time on the conveyor belt constitutes a dead-time in the level control loop.

2. The dead-time will be eliminated if the level controller adjusts both the screw speed and the conveyor belt speed, proportionally. This is illustrated in Figure 23.16.

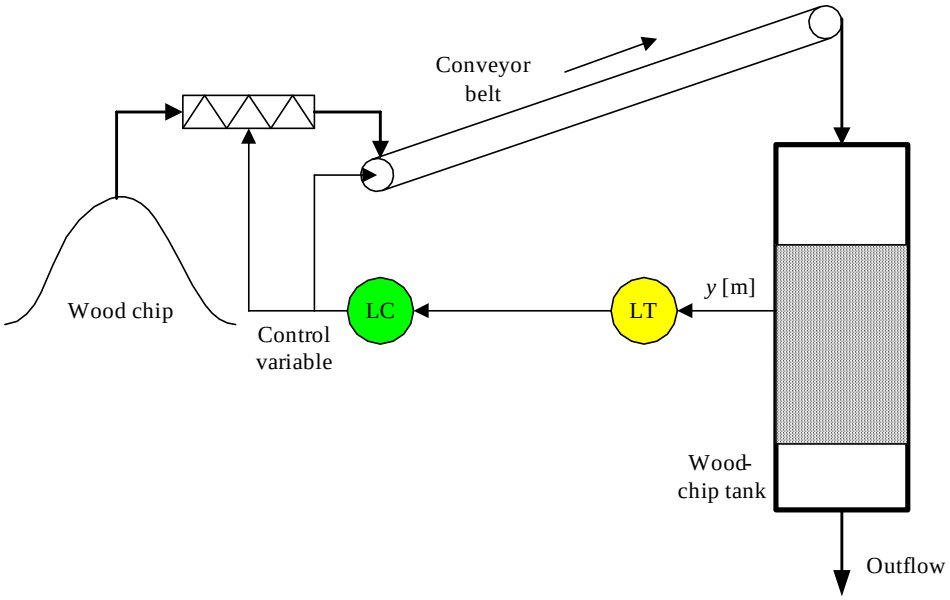

Figure 23.16:

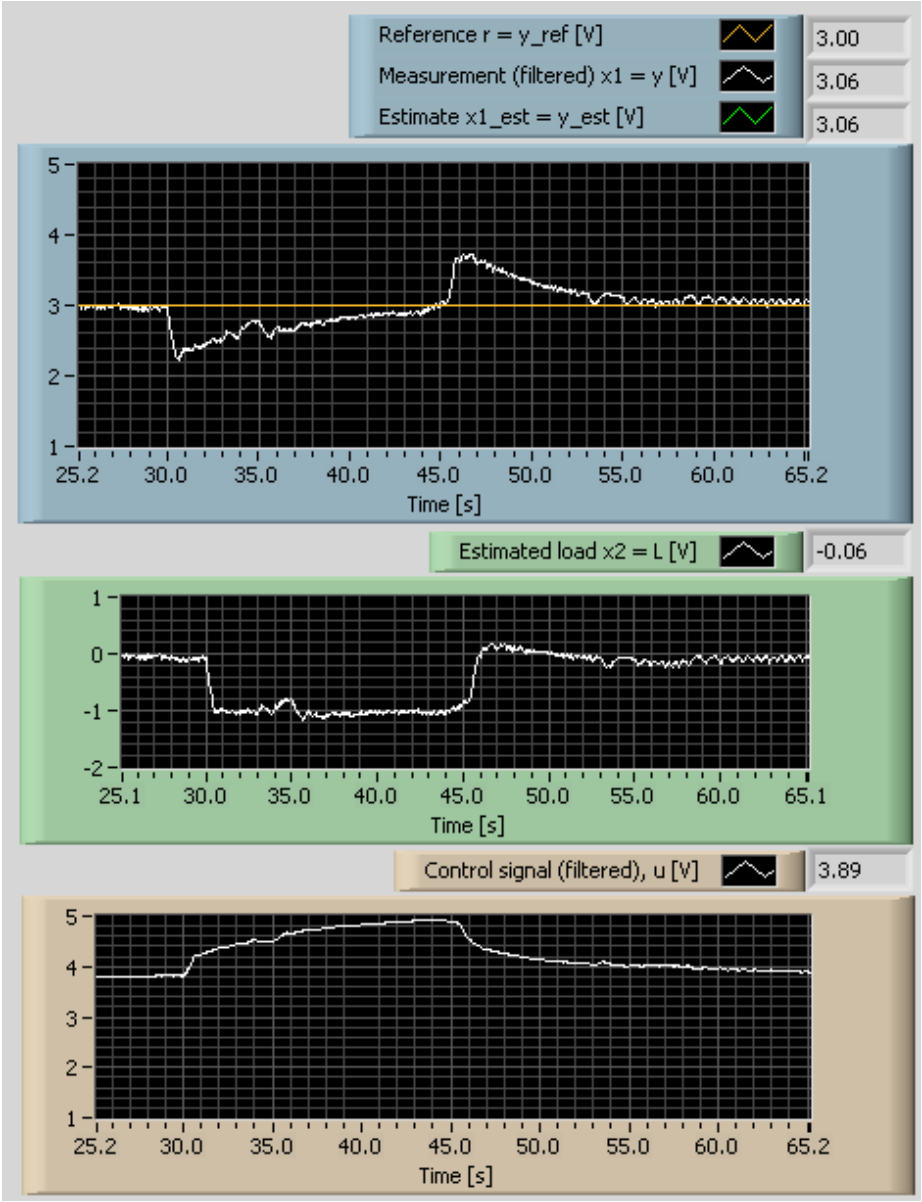
Reference r = y_ref [V]
3.00
Measurement (filtered) x1 = y [V]
3.06
Estimate x1_est = y_est [V]
3.06
5
4
3
2
1
25.2
30.0
35.0
40.0
45.0
50.0
55.0
60.0
65.2
Time [s]
Estimated load x2 = L [V]
-0.06
1
0
-1
-2
25.1
30.0
35.0
40.0
45.0
50.0
55.0
60.0
65.1
Time [s]
Control signal (filtered), u [V]
3.89
5
4
3
2
1
25.2
30.0
35.0
40.0
45.0
50.0
55.0
60.0
65.2
Time [s]

Figure 23.17:

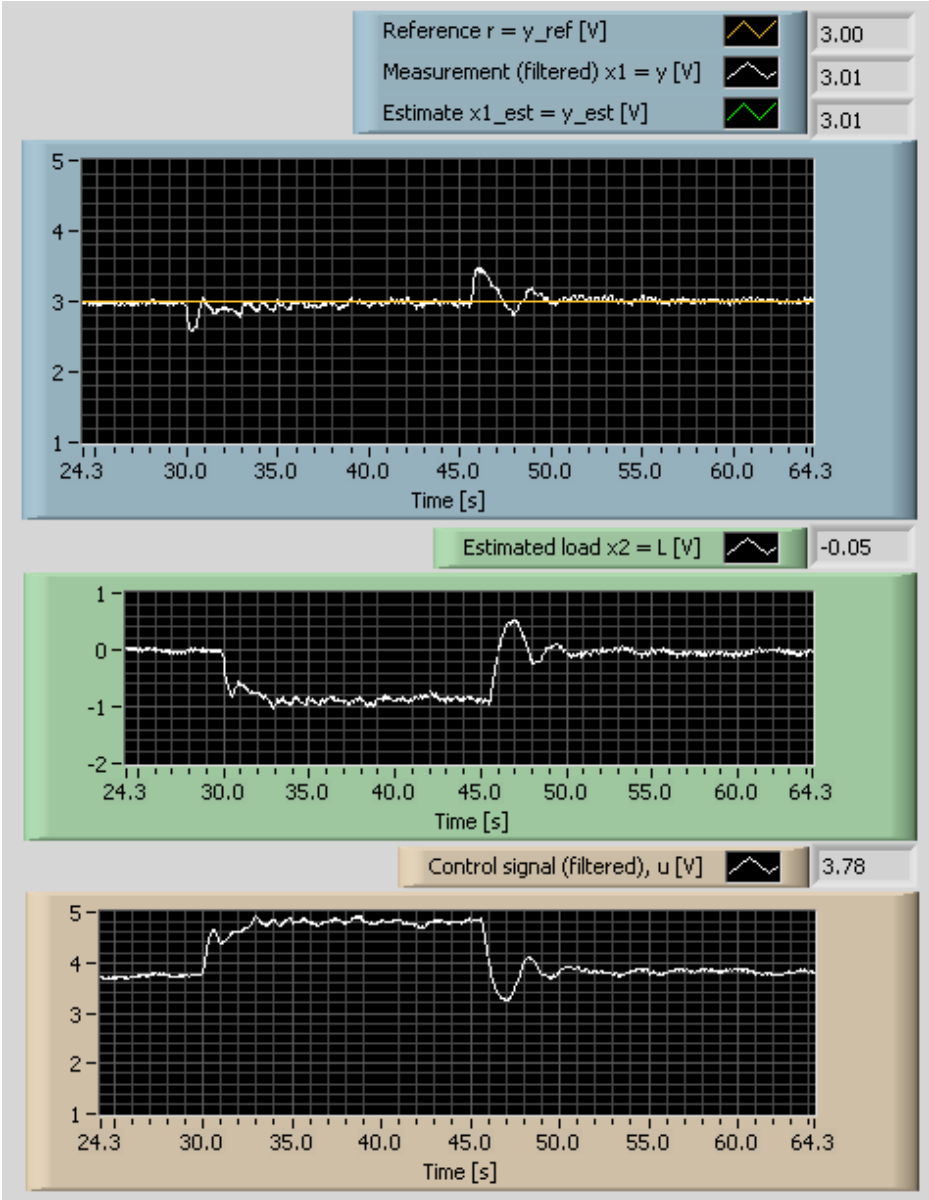
Reference r = y_ref [V]
3.00
Measurement (filtered) x1 = y [V]
3.01
Estimate x1_est = y_est [V]
3.01
5
4
3
2
1
24.3
30.0
35.0
40.0
45.0
50.0
55.0
60.0
64.3
Time [s]
Estimated load x2 = L [V]
-0.05
1
0
-1
-2
24.3
30.0
35.0
40.0
45.0
50.0
55.0
60.0
64.3
Time [s]
Control signal (filtered), u [V]
3.78
5
4
3
2
1
24.3
30.0
35.0
40.0
45.0
50.0
55.0
60.0
64.3
Time [s]

Figure 23.18:

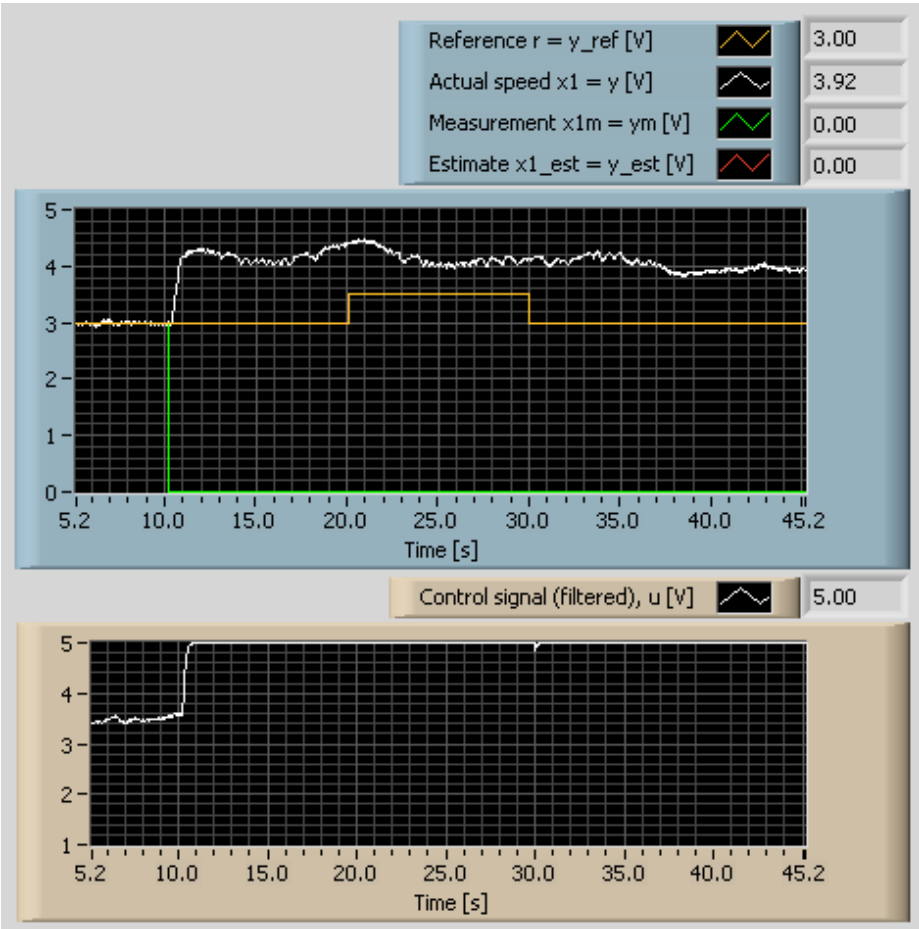
Reference r = y_ref [V]
3.00
Actual speed x1 = y [V]
3.92
Measurement x1m = ym [V]
0.00
Estimate x1_est = y_est [V]
0.00
5
4
3
2
1
0
5.2
10.0
15.0
20.0
25.0
30.0
35.0
40.0
45.2
Time [s]
Control signal (filtered), u [V]
5.00
5
4
3
2
1
5.2
10.0
15.0
20.0
25.0
30.0
35.0
40.0
45.2
Time [s]

Figure 23.19:

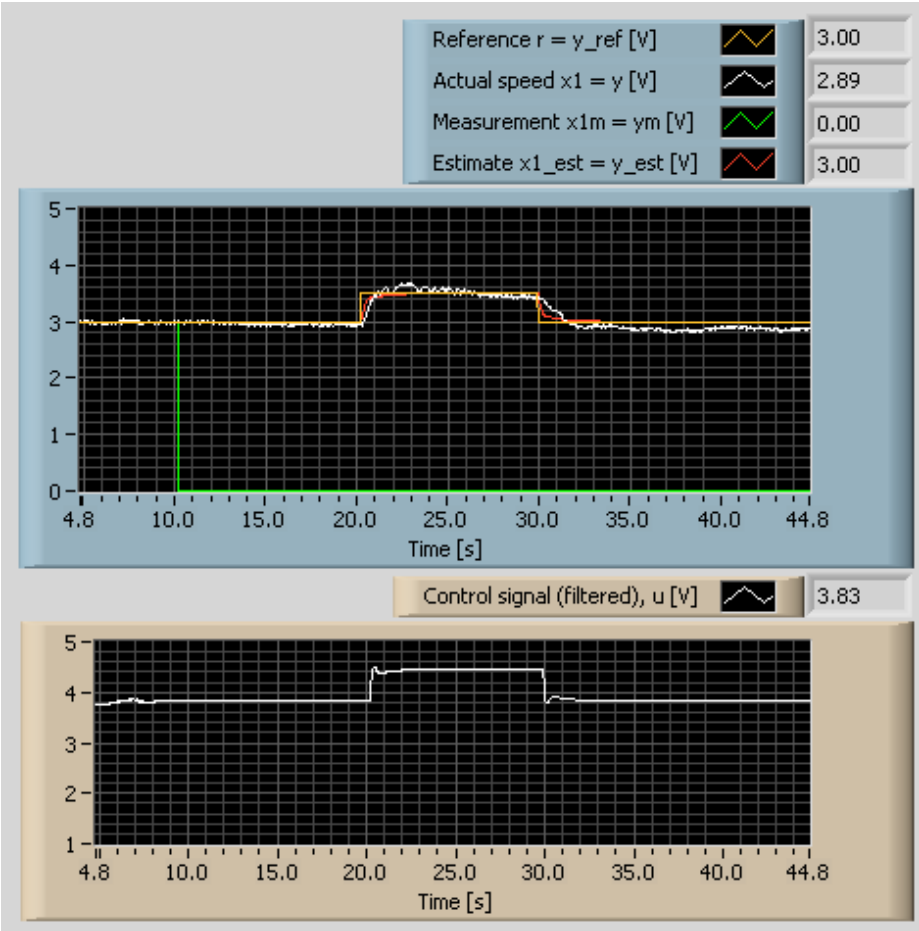
Reference r = y_ref [V]
3.00
Actual speed x1 = y [V]
2.89
Measurement x1m = ym [V]
0.00
Estimate x1_est = y_est [V]
3.00
5
4
3
2
1
0
4.8
10.0
15.0
20.0
25.0
30.0
35.0
40.0
44.8
Time [s]
Control signal (filtered), u [V]
3.83
5
4
3
2
1
4.8
10.0
15.0
20.0
25.0
30.0
35.0
40.0
44.8
Time [s]

Figure 23.20:

Appendix A

Models with parameter values

A.1 Electric motor

The motor is used in Exercises 19.1, 17.1, and 18.1.

Figure A.1 shows the electric motor (a DC motor). It is manipulated with

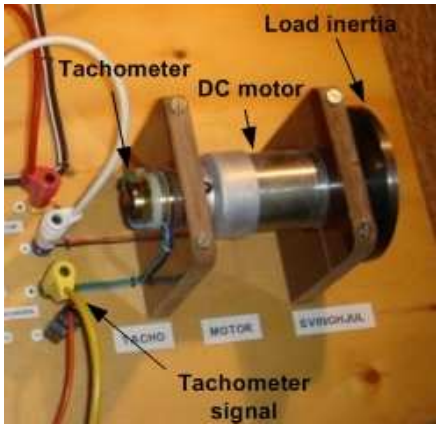

Figure A.1:

an input voltage signal, u, which is in the range $[-10\text{ V}, +10\text{ V}]$.

The rotational speed is measured with a tachometer which produces a output voltage signal which is proportional to the speed. The speed, S

[krpm = kilo revolutions per minute] is calculated continuously from the tachometer voltage, and hence the speed is assumed to be known at any instant of time. A proper mathematical model of the motor is

$$T_m \dot{S}(t) + S(t) = K_m[u(t) + L(t)] \tag{A.1}$$

L is equivalent load torque (represented in the same unit as the control variable, namely voltage). L can be regarded as a process disturbance. K_m is gain. T_m is time-constant. Parameter values which can be used in e.g. simulations are

$$T_m = 0.3 \text{ sec} \tag{A.2}$$

$$K_m = K = 0.17 \text{ krpm/V} \tag{A.3}$$

Figure A.2 shows a block diagram of the motor.

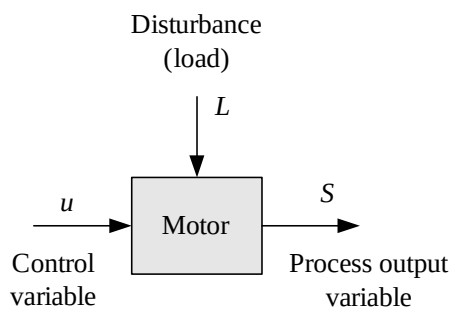

Figure A.2:

A.2 Ship

Figure 20.2 shows a ship.

The ship is used in Exercises 20.2, 21.1, and 22.2.

The parameter values presented in this appendix are realistic values of a specific ship.

Only the surge (forward-backward) direction is considered in the exercises in this book.

Applying Newtons's Law of Motion we obtain the following mathematical

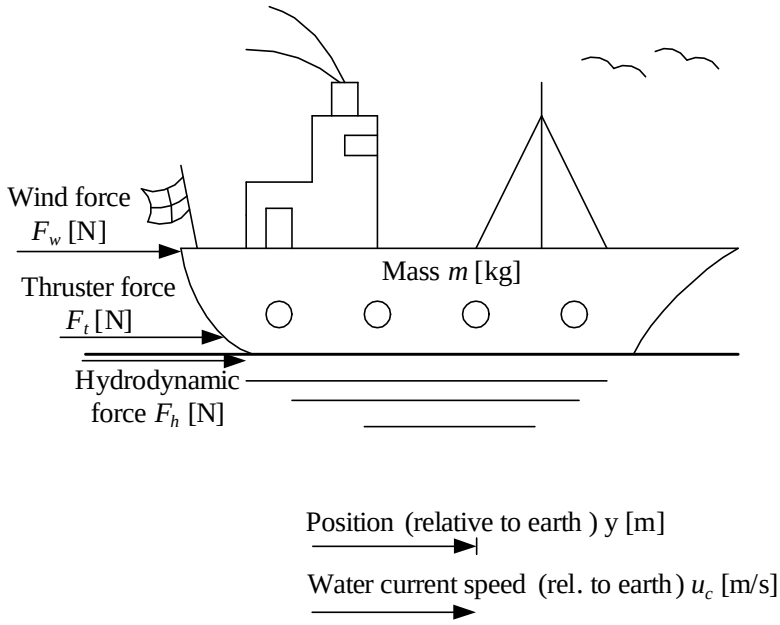

Figure A.3:

model of the surge motion:

$$m\ddot{y} = \underbrace{-D|\dot{y} - u_c|\,(\dot{y} - u_c)}_{F_h} + F_w(\phi, V_w) + F_t \tag{A.4}$$

Position is y [m]. Speed is $\dot{y}$ [m/s].

Thruster force (applied to move the ship) is F_t [N]. Maximum force is 552 kN forwards and 467 kN backwards.

Mass:

$$m = 71164 \text{ tons} = 71164 \cdot 10^3 \text{ kg} \tag{A.5}$$

Water current speed is u_c [m/s]. It typically varies in the range 0 – 3 m/s.

Hydrodynamic coeffiecient:

$$D = -8.4 \text{ kN}/\left[(\text{m/s})^2\right] \tag{A.6}$$

The wind force F_w acting on the ship is a function of the wind attack angle ϕ and the wind speed V_w [m/s]. The wind model is

$$F_w = V_w^2[c_1 \cos(\phi) + c_2 \cos(3\phi)] \text{ [kN]} \tag{A.7}$$

Wind coefficients:

$$c_1 = 0.1838 \tag{A.8}$$

$$c_2 = -0.0068 \tag{A.9}$$

Figure A.4 shows the wind scale.

m/s	Description
0.5 - 1.8 m/sec	light air
1.9 - 3.3 m/sec	light breeze
3.4 - 5.4 m/sec	gentle breeze
5.5 - 7.9 m/sec	breeze
8.0 - 11.0 m/sec	fresh breeze
11.1 - 14.1 m/sec	strong breeze
14.2 - 17.2 m/sec	near gale
17.3 - 20.8 m/sec	gale
20.9 - 24.4 m/sec	strong gale
24.5 - 28.5 m/sec	storm
28.6 - 32.6 m/sec	violent storm
> 32.6 m/sec	hurricane

Figure A.4:

Figure A.5 shows a block diagram of the ship.

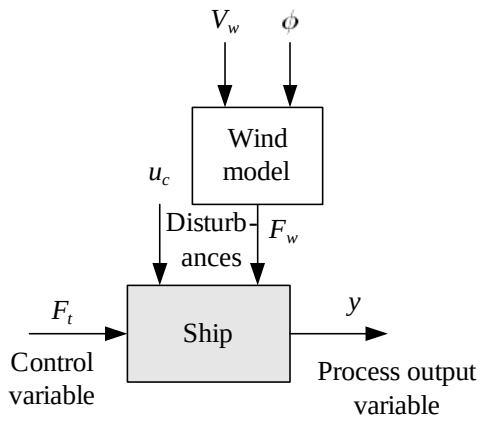
V_w
ϕ
Wind model
u_c
Disturb-ances
F_w
F_t
Ship
y
Control variable
Process output variable

Figure A.5: