Derivation of a Discrete-Time Lowpass Filter

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A lowpass filter is used to smooth out high frequent or random noise in a measurement signal. A very common lowpass filter in computer-based control systems is the discretized first order – or time-constant – filter. You can derive such a filter by discretizing the Laplace transfer function of the filter. A common discretization method in control applications is the (Euler) Backward Differentiation method. We will now derive a discrete-time filter using the Backward Differentiation method.

The Laplace transform transfer function – also denoted the continuous-time transfer function – of a first order lowpass filter is

\[ H(s) = \frac{y(s)}{u(s)} = \frac{1}{T_f s + 1} \]  

where \( T_f [s] \) is the time-constant. \( u \) is filter input, and \( y \) is filter output.

Cross-multiplying in (1) gives

\[(T_f s + 1) y(s) = u(s) \]  

Resolving the parenthesis gives

\[ T_f y(s) + y(s) = u(s) \]  

Taking the inverse Laplace transform of both sides of this equation gives the following differential equation (because multiplying by \( s \) means time-differentiation in the time-domain):

\[ T_f \dot{y}(t) + y(t) = u(t) \]  

Let us use \( t_k \) to represent the present point of time – or discrete time:

\[ T_f \dot{y}(t_k) + y(t_k) = u(t_k) \]  

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Here we will substitute the Backward Differentiation approximation for the time derivative. This approximation is

\[ \dot{y}(t_k) \approx \frac{y(t_k) - y(t_{k-1})}{h} \]  

(6)

It is illustrated in Figure 1.

\[ T_f \frac{y(t_k) - y(t_{k-1})}{h} + y(t_k) = u(t_k) \]  

(7)

Solving for \( y(t_k) \) gives

\[ y(t_k) = \frac{T_f}{T_f + h} y(t_{k-1}) + \frac{h}{T_f + h} u(t_k) \]  

(8)

which is commonly written as

\[ y(t_k) = (1 - a) y(t_{k-1}) + au(t_k) \]  

(9)

with filter parameter

\[ a = \frac{h}{T_f + h} \]  

(10)
which has a given value once you have specified the filter time-constant $T_f$ and the time-step $h$ is given. (9) is the formula for the filter output. It is ready for being programmed. This filter is frequently denoted the \textit{exponentially weighted moving average (EWMA) filter}, but we can just denote it a first order lowpass filter.

According to (9) the present filter output $y(t_k)$ is a function of the present filter input $u(t_k)$ and the filter output at the previous discrete time, $y(t_{k-1})$. Therefore, the filter output must be stored in the program so that it is available at the following execution if the filter algorithm.

It is important that the filter time-step $h$ is considerably smaller than the filter time-constant, otherwise the filter may behave quite differently from the original continuous-time filter (1) from which it is derived. $h$ should be selected as

$$h \leq \frac{T_f}{5} \quad (11)$$