Calculating Random Noise Attenuation Through a Lowpass Filter

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Given the following discrete-time lowpass filter:

$$y(t_k) = (1-a)\, y(t_{k-1}) + au(t_k) \tag{1}$$

If this filter stems from discretizing the continuous-time filter

$$H(s) = \frac{1}{T_f s + 1} \tag{2}$$

with the Euler Backward Difference method with time-step h [s], then

$$a = \frac{h}{T_f + h} \tag{3}$$

Inversely,

$$T_f = h\frac{1-a}{a} \tag{4}$$

We will now analyze the discrete-time lowpass filter (1). We assume that the filter input, u, is a random signal with standard deviation σ_u and hence variance

$$\mathrm{Var}(u) = \sigma_u^2 \tag{5}$$

We will now calculate the stationary variance of the filter output, y. This variance will be a function of the filter parameter a and the sampling interval h. This function will then be solved for a so that we get a formula a. In the following the following simplifying notation will be used:

$$x_k = x(t_k) \tag{6}$$

The variance is actually the expected value of the square of the signal value (we assume here that the mean value of the signal considered is zero):

$$\mathrm{Var}\left[y_k\right] = \sigma_y^2 = E(y_k^2) \tag{7}$$

From (1) the output variance is

$$
\begin{aligned}
E(y_k^2) &= E\left\{\left[(1-a)\,y_{k-1}+au_k\right]\left[(1-a)\,y_{k-1}+au_k\right]\right\} && (8)\\
&= E\left[(1-a)^2\,y_{k-1}^2\right]+E\left[2\,(1-a)\,y_{k-1}au_k\right]+E\left[a^2u_k^2\right] && (9)\\
&= (1-a)^2\underbrace{E\left[y_{k-1}^2\right]}_{=E\left[y_k^2\right]}+2\,(1-a)\,a\underbrace{E\left[y_{k-1}u_k\right]}_{=0}+a^2\underbrace{E\left[u_k^2\right]}_{=\sigma_u^2} && (10)\\
&= (1-a)^2\,E\left[y_k^2\right]+a^2\sigma_u^2 && (11)
\end{aligned}
$$

In (10) $E\left[y_{k-1}u_k\right]=0$ because y_{k-1} and u_k are uncorrelated (independent). In (10) $E\left[y_{k-1}^2\right]=E\left[y_k^2\right]$ because it is assumed that the signals are stationary. Solving (11) for $E(y_k^2)$ gives

$$
E(y_k^2)=\sigma_y^2=\frac{a^2}{1-(1-a^2)}\sigma_u^2=\frac{a^2}{(2a-a^2)}\sigma_u^2=\frac{a}{2-a}\sigma_u^2 \tag{12}
$$

Thus, the ratio between the output standard deviation and the input standard deviation is

$$
\frac{\sigma_y}{\sigma_u}=\sqrt{\frac{a}{2-a}}\stackrel{\text{def}}{=}K_\sigma \tag{13}
$$

where K_σ is the ratio between the output and input standard deviations. So, once a is given, K_σ can be calculated.

You can also go the opposite way: Solving (13) for a gives

$$
a=\frac{2K_\sigma^2}{1+K_\sigma^2} \tag{14}
$$

So, once you have specified K_σ, you know what filter parameter to use. If you want the filter time-constant T_f in stead of a, you can use (4).