Basic
DYNAMICS and CONTROL

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Chapter 4

The Laplace transform

4.1 Introduction

The Laplace transform is a mathematical tool which is useful in systems theory. It is the foundation of transfer functions which is a standard model form of dynamic systems. Transfer functions are described in Chapter 5. Furthermore, with the Laplace transform you relatively easily calculate responses in dynamic systems by hand.[1]

In this chapter I present the Laplace transform at a minimum level. You can find much more information in a mathematics text-book.

4.2 Definition of the Laplace transform

Given a time-evaluated function $f(t)$ – that is, $f(t)$ is a function of time t. It can be a sinusoid, a ramp, an impulse, a step, a sum of such functions, or any other function of time. The Laplace transform of $f(t)$ can be denoted $F(s)$, and is given by the following integral:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \qquad (4.1)$$

[1] It turns out that we rarely need to perform manual calculations of the responses. When we need to know the responses, it is in most situations more conventient to obtain them by simulating the system. With the Laplace transform you can calculate responses only for *linear systems*, that is, systems having a model which can be expressed as a linear differential equation.

Expressed with words, $f(t)$ is multiplied by the weight function e^{-st}, and the resulting product $e^{-st}f(t)$ is integrated from start time $t = 0$ to end time $t = \infty$. The Laplace transform does not care about any value that $f(t)$ might have at negative values of time t. In other words, you can think of $f(t)$ as being "switched on" at $t = 0$. (The so-called two-sided Laplace transform is defined also for negative time, but it is not relevant for our applications.)

s is the Laplace variable.[2] $F(s)$ is a function of s. The time t is not a variable in $F(s)$ – it disappeared through the time integration. $F(s)$ will look completely different from $f(t)$, cf. the following example.

Example 4.1 ***Laplace transform of a step***

Given the function

$$f(t) = 1 \text{ (for } t \geq 0) \tag{4.2}$$

which is a step of amplitude 1 at time $t = 0$. Using (4.1), its Laplace transform becomes

$$F(s) = \int_0^\infty e^{-st} \cdot 1 \cdot dt = \left[-\frac{1}{s} e^{-st} \right]_{t=0}^{t=\infty} = \frac{1}{s} \tag{4.3}$$

[End of Example 4.1]

Calculating the time function $f(t)$ from its Laplace transform $F(s)$ – in other words: going from s to t – is denoted *inverse Laplace transform*. This can be expressed as

$$f(t) = \mathcal{L}^{-1} \{F(s)\} \tag{4.4}$$

The inverse Laplace transform is actually defined by a complicated complex integral.[3] If you *really* want to calculate this integral, you should use the Residue Theorem in mathematics. However, I suggest you in stead try the simplest method, namely to find $f(t)$ from the precalculated Laplace transform pairs, cf. Section 4.3, possibly combined with one or more of the Laplace transform properties, cf. Section 4.4.

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[2] You may wonder what is the physical meaning of s. It can be interpreted as a complex frequency, but I think the best answer is that there is no meaningful physical meaning.

[3] $f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$

4.3 Laplace transform pairs

$F(s)$ and $f(t)$ in Example 4.1 can be denoted a *Laplace transform pair*:

$$F(s) = \frac{1}{s} \quad \Longleftrightarrow \quad f(t) = 1 \qquad \text{(step of amplitude 1)} \tag{4.5}$$

Below are some additional Laplace transform pairs. Each time function, $f(t)$, is defined for $t \geq 0$. $F(s)$ can be derived from (4.1). In the expressions below, k is some constant. For example, (4.5) is (4.7) with $k = 1$.

Laplace transform pairs:

$$F(s) = k \quad \Longleftrightarrow \quad f(t) = k\delta(t) \qquad \text{(impulse of strength or area } k\text{)} \tag{4.6}$$

$$\frac{k}{s} \quad \Longleftrightarrow \quad k \qquad \text{(step of amplitude } k\text{)} \tag{4.7}$$

$$\frac{k}{s^2} \quad \Longleftrightarrow \quad kt \qquad \text{(ramp of slope } k\text{)} \tag{4.8}$$

$$k\frac{n!}{s^{n+1}} \quad \Longleftrightarrow \quad kt^n \tag{4.9}$$

$$\frac{k}{Ts+1} \quad \Longleftrightarrow \quad \frac{ke^{-t/T}}{T} \tag{4.10}$$

$$\frac{k}{(Ts+1)s} \quad \Longleftrightarrow \quad k\left(1 - e^{-t/T}\right) \tag{4.11}$$

$$\frac{k}{(T_1 s+1)(T_2 s+1)} \quad \Longleftrightarrow \quad \frac{k}{T_1 - T_2}\left(e^{-t/T_1} - e^{-t/T_2}\right) \tag{4.12}$$

$$\frac{k}{(T_1 s+1)(T_2 s+1)s} \quad \Longleftrightarrow \quad k\left[1 + \frac{1}{T_2 - T_1}\left(T_1 e^{-t/T_1} - T_2 e^{-t/T_2}\right)\right] \tag{4.13}$$

4.4 Laplace transform properties

In calculations with the Laplace transform you will probably need one or more of the Laplace transform *properties* presented below.[4] We will definitely use some of them for deriving transfer functions, cf. Chapter 5. Each of these properties can be derived from the Laplace transform definition (4.1).

Linear combination:

$$k_1F_1(s) + k_2F_2(s) \Longleftrightarrow k_1f_1(t) + k_2f_2(t) \tag{4.14}$$

Special case: Multiplication by a constant:

$$kF(s) \leftrightarrow kf(t) \tag{4.15}$$

Time delay:

$$F(s)e^{-\tau s} \quad \Longleftrightarrow \quad f(t-\tau) \tag{4.16}$$

Time derivative:

$$s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \ldots - \overset{(n-1)}{f}(0) \quad \Longleftrightarrow \quad \overset{(n)}{f}(t) \tag{4.17}$$

Special case: Time derivative with zero initial conditions:

$$s^nF(s) \quad \Longleftrightarrow \quad \overset{(n)}{f}(t) \tag{4.18}$$

Special case: Time derivative with non-zero initial condition:

$$sF(s) - f_0 \quad \Longleftrightarrow \quad \dot{f}(t) \tag{4.19}$$

Special case: First order time derivative with zero initial condition:

$$sF(s) \quad \Longleftrightarrow \quad \dot{f}(t) \tag{4.20}$$

(So, differentiation corresponds to multiplication by s.)

[4] Additional properties could have been given here, too, but the ones presented are the most useful.

Integration:

$$\frac{1}{s}F(s) \iff \int_0^t f(\tau)d\tau \tag{4.21}$$

(So, integration corresponds to division by s.)

Final Value Theorem:

$$\lim_{s\to 0} sF(s) \iff \lim_{t\to\infty} f(t) \tag{4.22}$$

Example 4.2 ***Calculation of time response (inverse Laplace transform)***

Given the following differential equation:

$$\dot{y}(t) = -2y(t) \tag{4.23}$$

with initial value $y(0) = 4$. Calculate $y(t)$ using the Laplace transform.

To calculate $y(t)$ we start by taking the Laplace transform of both sides of the differential equation (4.23):

$$\mathcal{L}\left\{\dot{y}(t)\right\} = \mathcal{L}\left\{-2y(t)\right\} \tag{4.24}$$

Here, we apply the time derivative property (4.19) at the left side, and the linear combination property (4.15) to the right side, to get

$$sY(s) - 4 = -2Y(s) \tag{4.25}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{4}{s+2} \tag{4.26}$$

To get the corresponding $y(t)$ from this $Y(s)$ we look for a proper Laplace transform pair. (4.10) fits. We have to write our $Y(s)$ on the form of (4.10). Dividing both the numerator and the denominator by 2 gives

$$Y(s) = \frac{4}{s+2} = \frac{2}{0.5s+1} = \frac{k}{Ts+1} \tag{4.27}$$

Hence, $k = 2$ and $T = 0.5$. Finally, according to (4.10) $y(t)$ becomes

$$\underline{\underline{y(t)}} = \frac{ke^{-t/T}}{T} = \frac{2e^{-t/0.5}}{0.5} = \underline{\underline{4e^{-2t}}} \tag{4.28}$$

[End of Example 4.2]

Example 4.3 ***Calculation of steady-state value using the Final Value Theorem***

See Example 4.2. The steady-state value of y in (4.28) is

$$y_s = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} 4e^{-2t} = 0 \tag{4.29}$$

Using the Final Value Theorem we get

$$y_s = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{4}{s+2} = 0 \tag{4.30}$$

So, the results are the same.

[End of Example 4.3]

Chapter 5

Transfer functions

5.1 Introduction

Transfer functions is a model form based on the Laplace transform, cf. Chapter 4. Transfer functions are very useful in analysis and design of linear dynamic systems, in particular controller functions and signal filters. The main reasons why transfer functions are useful are as follows:

- **Compact model form**: If the original model is a higher order differential equation, or a set of first order differential equations, the relation between the input variable and the output variable can be described by one transfer function, which is a rational function of the Laplace variable s, without any time-derivatives.
- **Representation of standard models**: Transfer functions are often used to represent standard models of controllers and signal filters.
- **Simple to combine systems**: For example, the transfer function for a combined system which consists of two systems in a series combination, is just the product of the transfer functions of each system.
- **Simple to calculate time responses**: The calculations will be made using the Laplace transform, and the necessary mathematical operations are usually much simpler than solving differential equations. Calculation of time-responses for transfer function models is described in Chapter 5.5.
- **Simple to find the frequency response**: The frequency response is a function which expresses how sinusoid signals are transfered

through a dynamic system. Frequency response is an important tool in analysis and design of signal filters and control systems. The frequency response can be found from the transfer function of the system. However, frequency response theory is not a part of this book (a reference is [2]).

Before we start, I must say something about the mathematical notation: In the following sections, and in the reminder of the book, I use the *same symbol* (letter) for the time function, say $y(t)$, as for the Laplace transform of $y(t)$, here $y(s)$ – although it is mathematically incorrect to do it. The reason is to simplify the presentation. Now, only one variable name (symbol) is needed for both the Laplace domain and the time domain. For example, assume that $y(t)$ is the time function of the level y in a water tank. Then I write $y(s)$, although I formally should have written $Y(s)$ or $y^*(s)$ or $\overline{y}(s)$ (or something else that is different from $y(s)$) for $\mathcal{L}\{y(t)\}$. It is my experience (from many years together with transfer functions) that this simplifying notation causes no problems.

5.2 Definition of the transfer function

The first step in deriving the transfer function of a system is taking the Laplace transform of the differential equation (which must be linear). Let us go on with an example, but the results are general. Given the following mathematical model having two input variables u_1 and u_2 and one output variable y. (I think you will understand from this example how to find the transfer function for systems with different number of inputs and outputs.)

$$\dot{y}(t) = ay(t) + b_1u_1(t) + b_2u_2(t) \tag{5.1}$$

a, b_1 and b_2 are model parameters (coefficients). Let the initial state (at time $t = 0$) be y_0. We start by taking the Laplace transform of both sides of the differential equation:

$$\mathcal{L}\{\dot{y}(t)\} = \mathcal{L}\{ay(t) + b_1u_1(t) + b_2u_2(t)\} \tag{5.2}$$

By using the linearity property of the Laplace transform, cf. (4.14), the right side of (5.2) can be written as

$$\mathcal{L}\{ay(t)\} + \mathcal{L}\{b_1u_1(t)\} + \mathcal{L}\{b_2u_2(t)\} \tag{5.3}$$

$$= a\mathcal{L}\{y(t)\} + b_1\mathcal{L}\{u_1(t)\} + b_2\mathcal{L}\{u_2(t)\} \tag{5.4}$$

$$= ay(s) + b_1u_1(s) + b_2u_2(s) \tag{5.5}$$

The left side of (5.2) will be Laplace transformed using the differentiation rule, cf. (4.17), on $\mathcal{L}\{\dot{y}(t)\}$:

$$\mathcal{L}\{\dot{y}(t)\} = sy(s) - y_0 \tag{5.6}$$

Thus, we have found that the Laplace transformed (5.2) is

$$sy(s) - y_0 = ay(s) + b_1 u_1(s) + b_2 u_2(s) \tag{5.7}$$

Solving for the output variable $y(s)$ gives

$$y(s) = \overbrace{\frac{1}{s-a} y_0}^{y_{\text{init}}(s)} + \overbrace{\underbrace{\frac{b_1}{s-a}}_{H_1(s)} u_1(s)}^{y_1(s)} + \overbrace{\underbrace{\frac{b_2}{s-a}}_{H_2(s)} u_2(s)}^{y_2(s)} \tag{5.8}$$

In (5.8),

- y_1 is the contribution from input u_1 to the total response y,
- y_2 is the contribution from input u_2 to the total response y,
- y_{init} is the contribution from the initial state y_0 to the total response y.

Of course, these contributions to the total response are in the Laplace domain. The corresponding responses in the time domain are found by calculating the inverse Laplace transforms.

Now we have the following two *transfer functions* for our system:

- The transfer function from u_1 to y is

$$H_1(s) = \frac{b_1}{s-a} \tag{5.9}$$

- The transfer function from u_2 to y is

$$H_2(s) = \frac{b_2}{s-a} \tag{5.10}$$

Thus, *the transfer function from a given input variable to a given output variable is the s-valued function with which the Laplace transformed input variable is multiplied to get its contribution to the response in the output*

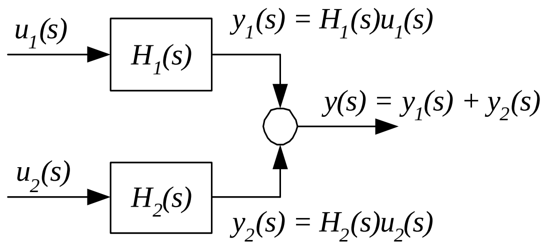

Figure 5.1: Block diagram representing the transfer functions $H_1(s)$ and $H_2(s)$ in (5.8)

variable. In other words: The transfer function expresses how the input variable is transferred through the system.

The transfer functions derived above can be illustrated with the block diagram shown in Figure 5.1.

One alternative way to express the definition of transfer function

From (5.8) we have

$$H_1(s) = \frac{b_1}{s-a} = \frac{y_1(s)}{u_1(s)} \tag{5.11}$$

So, we can define the transfer functions as the *ratio* between the Laplace transformed contribution to the total response in the output variable, here $y_1(s)$, and the Laplace transformed input variable, here $u_1(s)$. We may also say that the transfer functions is the ratio between the Laplace transformed total response in the output variable, here $y(s)$, and the Laplace transformed input variable, here $u_1(s)$, when all other inputs are set to zero and the initial state is zero.

5.3 Characteristics of transfer functions

A transfer function can be written on a factorized form – often called a *zero-pole form*:

$$H(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_r)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = \frac{b(s)}{a(s)} \tag{5.12}$$

Here $z_1, \ldots, z_r$ are the *zeros* and $p_1, \ldots, p_n$ are the *poles* of the transfer function. For example, the transfer function

$$H(s) = \frac{4s - 4}{s^2 + 5s + 6} = 4\frac{s - 1}{(s + 3)(s + 2)} \tag{5.13}$$

have two poles, -3 and -2, one zero, 1, and the gain is 4. (As shown in e.g. [2] the values of the poles determines the stability property of a system. The system is stable only if all the poles have negative real parts, in other words if all the poles lie in the left half part of the complex plane. But we will not go into any stability analysis here.)

The s-polynomial in the denominator of $H(s)$, which is $a(s)$ in (5.12), is denoted the *characteristic polynomial.* The poles are the roots of the characteristic polynomial, that is

$$a(s) = 0 \text{ for } s = s_1, s_2, ..., s_n \text{ (the poles)} \tag{5.14}$$

The *order* of a transfer function is the order of the characteristic polynomial. For example, the transfer function (5.13) has order 2.

5.4 Combining transfer functions blocks in block diagrams

Transfer function blocks may be combined in a block diagram according to the rules shown in Figure 5.2. One possible purpose of such a combination is to simplify the block diagram, or to calculate the resulting or overall transfer function. For example, the combined transfer function of two transfer functions connected in series is equal to the product of the individual transfer functions, jc. the Series connection rule in Figure 5.2.

5.5 How to calculate responses from transfer function models

It is quite easy to calculate responses in transfer function models manually (by hand). Assume given the following transfer function model:

$$y(s) = H(s)u(s) \tag{5.15}$$

To calculate the time-response $y(t)$ for a given input signal $u(t)$, we can do as follows:

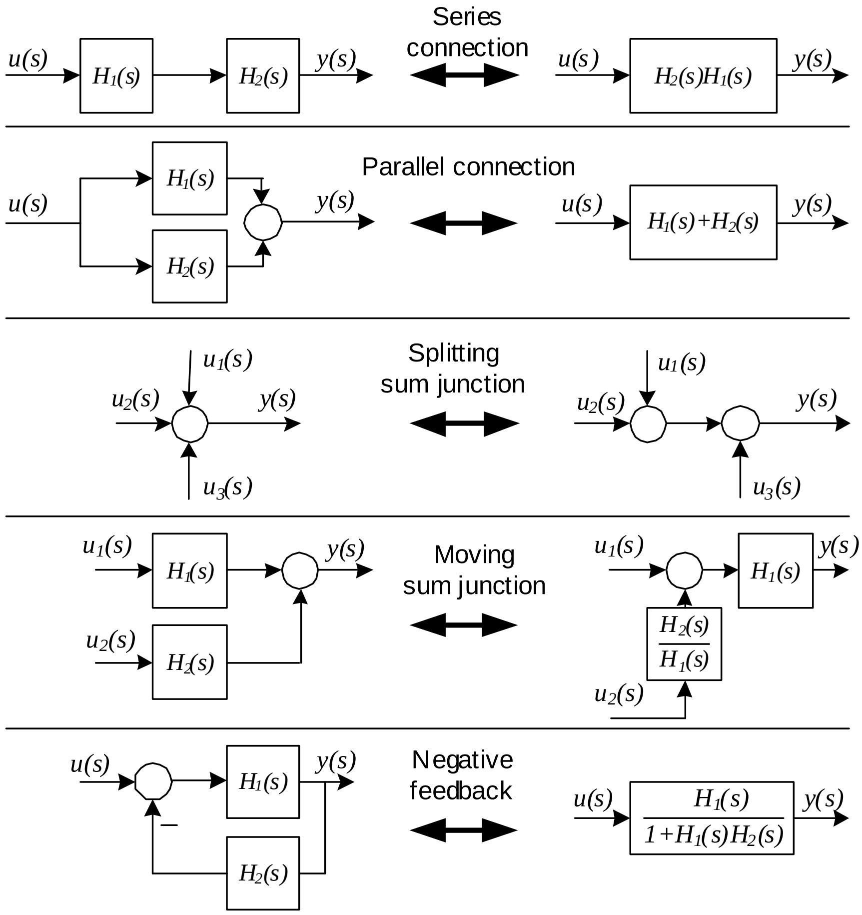

Figure 5.2: Rules for combining transfer function blocks

1. First, find $u(s)$ – the Laplace transform of the input signal. $u(s)$ can be found from precalculated Laplace transform pairs, cf. Section 4.3, possibly combined with one or more of the Laplace transform properties, cf. Section 4.4, where particularly the linearity property (4.14) is useful.

2. The Laplace transform of the output signal, $y(s)$, is calculated from (5.15), that is,

$$y(s) = H(s)u(s) \tag{5.16}$$

where $u(s)$ is found as explained above.

3. The time-function $y(t)$ is calculated as the inverse Laplace transform of $y(s)$, cf. Chapter 4.

Example 5.1 ***Calculation of time-response for transfer function model***

Given the transfer function model

$$y(s) = \underbrace{\frac{3}{s+0.5}}_{H(s)} u(s) \tag{5.17}$$

Suppose that $u(t)$ is a step from 0 to 2 at $t = 0$. We shall find an expression for the time-response $y(t)$. The Laplace transform of $u(t)$ is, cf. (4.7),

$$u(s) = \frac{2}{s} \tag{5.18}$$

Inserting this into (5.17) gives

$$y(s) = \frac{3}{s+0.5} \cdot \frac{2}{s} = \frac{6}{(s+0.5)s} = \frac{12}{(2s+1)s} \tag{5.19}$$

(5.19) has the same form as the Laplace transform pair (4.11) which is repeated here:

$$\frac{k}{(Ts+1)s} \iff k\left[1 - e^{-t/T}\right] \tag{5.20}$$

Here $k = 12$ and $T = 2$. The time-response becomes

$$y(t) = 12\left[1 - e^{-t/2}\right] \tag{5.21}$$

Figure 5.3 shows $y(t)$. The steady-state response is 12, which can be calculated from $y(t)$ by setting $t = \infty$.

[End of Example 5.1]

5.6 Static transfer function and static response

Suppose that the input signal to a system is a step of amplitude u_s. The corresponding static time-response can found from the Final Value Theorem:

$$y_s = \lim_{s\to 0} s \cdot y(s) = \lim_{s\to 0} s \cdot H(s)\frac{u_s}{s} = \underbrace{\lim_{s\to 0} H(s)}_{H_s} u_s \tag{5.22}$$

where H_s is the *static transfer function.* That is,

$$H_s = \lim_{s\to 0} H(s) \tag{5.23}$$

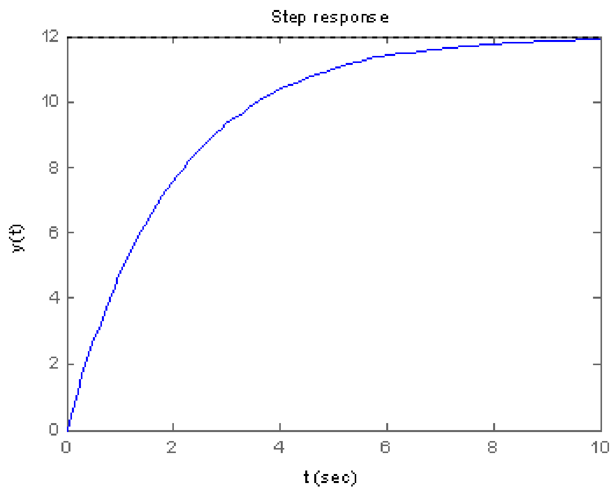

Figure 5.3: Example 5.1: The time-response $y(t)$ given by (5.21)

Thus, the static transfer function, H_s, is found by letting s approach zero in the transfer function, $H(s)$.

Once we know the static transfer function H_s the static (steady-state) response y_s due to a constant input of value u_s, is

$$y_s = H_s u_s \tag{5.24}$$

Example 5.2 ***Static transfer function and static response***

See Example **??**. The transfer function is

$$H(s) = \frac{3}{s + 0.5} \tag{5.25}$$

The corresponding static transfer function becomes

$$H_s = \lim_{s \to 0} H(s) = \lim_{s \to 0} \frac{3}{s + 0.5} = 6 \tag{5.26}$$

Assume that the input u has the constant value of $u_s = 2$. What is the corresponding static response y_s in the output? It can be calculated from the static transfer function as

$$y_s = H_s u_s = 6 \cdot 2 = 12 \tag{5.27}$$

which is the same result as found in Example 5.1.

[End of Example 5.2]

Chapter 6

Dynamic characteristics

6.1 Introduction

In this chapter a number of standard dynamic models in the form of transfer functions will be defined. With such standard models you can characterize the dynamic properties of a physical system in terms of for example gain, time-constant, and time-delay. These terms are also useful for controller tuning, as in the Skogestad's tuning method which is described in Section 10.3.

This chapter covers integrators, time-constant systems and time-delays. Second order systems, which may show oscillatory responses, are not covered.[1]

6.2 Integrators

An integrator is a system where *the output variable y is the time integral of the input variable u*, multiplied with some gain K:

$$y(t) = K \int_0^t u \, d\theta \tag{6.1}$$

[1]Because I have found that the theory about second order systems is not important in most applications. However, an article about second order systems is available at http://techteach.no.

Taking the time derivative of both sides of (6.1) yields the following differential equation describing an integrator:

$$\dot{y}(t) = Ku(t) \tag{6.2}$$

Taking the Laplace transform using (4.20) gives

$$sy(s) = Ku(s) \tag{6.3}$$

which gives the following *transfer function* of an integrator:

$$H(s) = \frac{y(s)}{u(s)} = \frac{K}{s} \tag{6.4}$$

Let us now find the *step response* of the integrator. We assume that $u(t)$ is a step of amplitude U at $t = 0$. From (4.7) $u(s) = \frac{U}{s}$. Thus,

$$y(s) = H(s)u(s) = \frac{K}{s} \cdot \frac{U}{s} = \frac{KU}{s^2} \tag{6.5}$$

which, inverse Laplace transformed using (4.8), is

$$y(t) = KUt \tag{6.6}$$

Thus, the step response of an integrator is *a ramp* with rate KU. Figure 6.1 shows simulated response of an integrator.

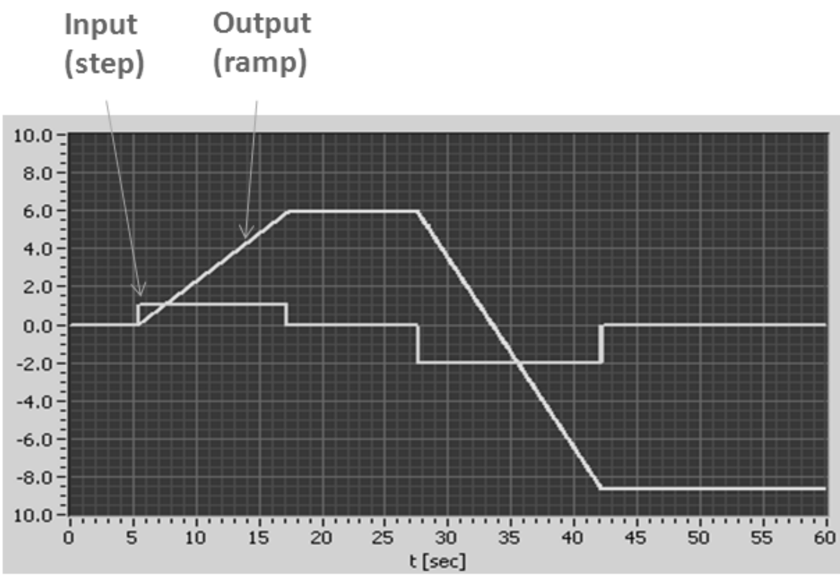

Figure 6.1: Simulated response of an integrator

Example 6.1 ***An integrator: A liquid tank***

See Example 3.1 on page 36 which describes a liquid tank. Assume for simplicity that there is no outflow from the tank. The mathematical model of this system is then

$$\dot{h}(t) = \frac{1}{A} q_i(t) \tag{6.7}$$

Taking the Laplace transform of (6.7) gives

$$sh(s) - h_0 = \frac{1}{A} q_i(s) \tag{6.8}$$

which gives

$$h(s) = \frac{h_0}{s} + \underbrace{\frac{1}{As}}_{H(s)} q_i(s) \tag{6.9}$$

So, the transfer function is

$$H(s) = \frac{h(s)}{q_i(s)} = \frac{1}{A} \cdot \frac{1}{s} \tag{6.10}$$

The system is an integrator!

It is actually quite naturally that the liquid tank is an integrator, since the level is proportional to the integral of the inflow. This can be seen by integrating (6.7), which gives

$$h(t) = h(0) + \int_0^t \frac{1}{A} q_i(\theta)\, d\theta \tag{6.11}$$

Whenever you need a concrete example of an integrator, recall the tank!

[End of Example 6.1]

6.3 Time-constant systems

In *time-constant* system – also denoted *first order systems* – the output variable y and the input variable u are related according to the following differential equation:

$$T\dot{y}(t) + y(t) = Ku(t) \tag{6.12}$$

Here K is the *gain,* and T is the *time-constant.*.

Taking the Laplace transform of both sides of (6.12) gives

$$y(s) = \underbrace{\frac{K}{Ts+1}}_{H(s)} u(s) \tag{6.13}$$

where $H(s)$ is the transfer function. Here is an example:

$$H(s) = \frac{3}{4s+2} = \frac{1.5}{2s+1} \tag{6.14}$$

The gain is $K = 1.5$, and the time-constant is $T = 2$ (in a proper time unit, e.g. seconds).

Let us study the *step response* of a first order system. We assume that the input signal $u(t)$ is a step of amplitude U at time $t = 0$. From (4.7) $u(s) = \frac{U}{s}$. The Laplace transformed response becomes

$$y(s) = H(s)u(s) = \frac{K}{Ts+1} \cdot \frac{U}{s} \tag{6.15}$$

Taking the inverse Laplace transform using (4.11) gives

$$y(t) = KU(1 - e^{-\frac{t}{T}}) \tag{6.16}$$

Let us study the importance of the parameters K and T for the step response. Its is assumes that the system is in an operating point where the input is u_0 and the ouput is y_0 before the step in the input u. Figure 6.2 shows a simulation of a first order system. The parameters used in the simulation are $U = 1$, $K = 2$, and $T = 5$.

- **Importance of** K: The steady-state response due to the input step is

  $$y_s = \lim_{t\to\infty} y(t) = KU \tag{6.17}$$

  which is found from (6.16) with $t \to \infty$. Thus, the step is amplified with the gain K in steady-state. This is confirmed in Figure 6.2.

  In Section 5.6 the static transfer function H_s was defined. What is the H_s of a time-constant system? We get

  $$H_s = \frac{y_s}{u_s} = \frac{KU}{U} = K \tag{6.18}$$

  So, the H_s is equal to the gain parameter, K.

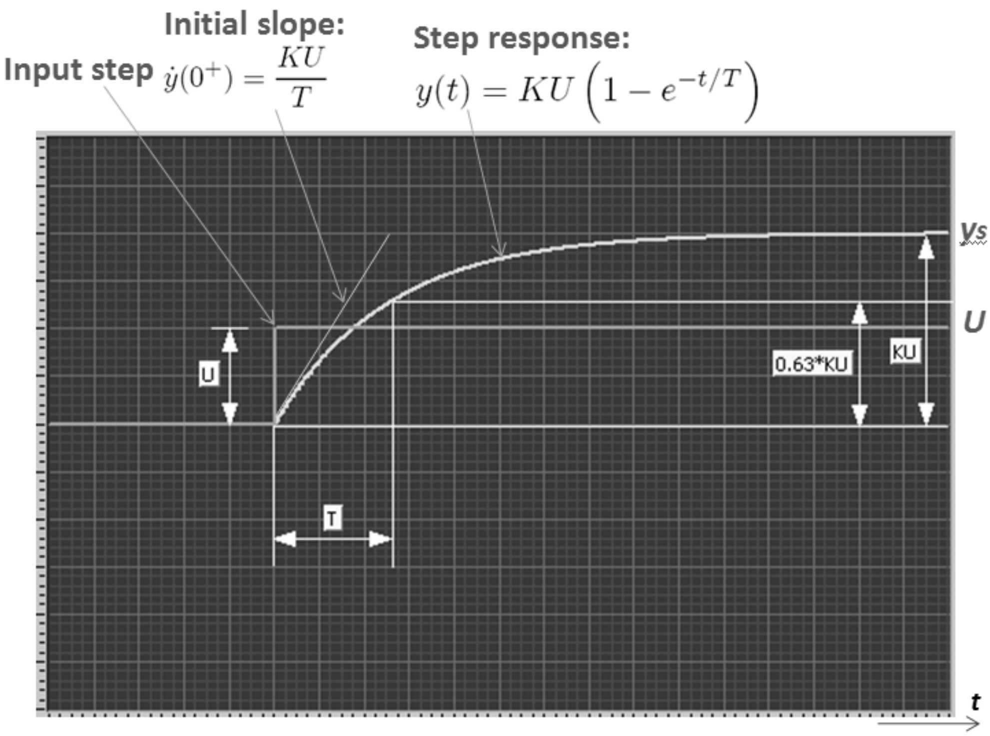

Figure 6.2: Step response of a time-constant system

- **Importance of** T: Let us set $t = T$ in (6.16):

$$y(T) = KU(1 - e^{-\frac{T}{T}}) \quad (6.19)$$

$$= KU(1 - e^{-1}) \quad (6.20)$$

$$= 0.63 \cdot KU \quad (6.21)$$

Thus, at time $t = T$ the step response has increased to 63% of the total increase which is KU. This is confirmed in Figure 6.2. This suggests a practical way to read off the time-constant from a step response curve.

Qualitatively, we can state the importance of the time-constant as follows: *The less T, the faster the system.*

Does the steady-state response depend on the time-constant? No, because the steady-state response is equal to $y_s = KU$ which is not dependent of T.

Example 6.2 ***First order system: Heated liquid tank***

In Example 3.3 on page 42 we developed a mathematical model of heated liquid tank (a thermal system). The model is repeated here:

$$cm\dot{T} = P + cF\,(T_i - T) + U_h(T_e - T) \tag{6.22}$$

Let's for simplicity assume that the tank is well isolated so that

$$U_h \approx 0 \tag{6.23}$$

We will now calculate the transfer functions from P to T and from T_i to T. Taking the Laplace transform of (6.22) gives

$$cm\,[sT(s) - T_0] = P(s) + cF\,[T_i(s) - T(s)] \tag{6.24}$$

Since we are to find the transfer function, we may set the initial value to zero:

$$T_0 = 0 \tag{6.25}$$

From (6.24) we will find

$$T(s) = \underbrace{\frac{K_1}{T_1 s + 1}}_{H_1(s)} P(s) + \underbrace{\frac{K_2}{T_1 s + 1}}_{H_2(s)} T_i(s) \tag{6.26}$$

The gains and the time-constant of each of the two transfer functions are

$$K_1 = \frac{1}{cF} \tag{6.27}$$

$$K_2 = 1 \tag{6.28}$$

$$T_1 = \frac{m}{F} = \frac{\text{Mass}}{\text{Flow}} \tag{6.29}$$

Comments:

- The time-constant, which represents the "dynamics", is the same for both transfer functions $H_1(s)$ and $H_2(s)$.
- In many applications the flow F may change. Assume that the flow is decreased. The dynamic properties of the system then change:
  - According to (6.27) the gain from P to T increases, and hence the T becomes more sensitive to P, giving higher value of T for a given change of P.
  - According to (6.29) the time-constant increases, causing a more sluggish response in T to a change in P.

[End of Example 6.2]

6.4 Time-delays

In many systems there is a *time-delay* or dead-time in the signal flow, for example with material transport on a conveyor belt, see Figure 6.3. In this application, the relation between the input variable F_{in} and the output variable F_{out} is

$$F_{out}(t) = F_{in}(t - \tau) \tag{6.30}$$

where τ is the time-delay which is the transportation time on the belt. In other words: The outflow at time t is equal to the inflow τ time units ago.

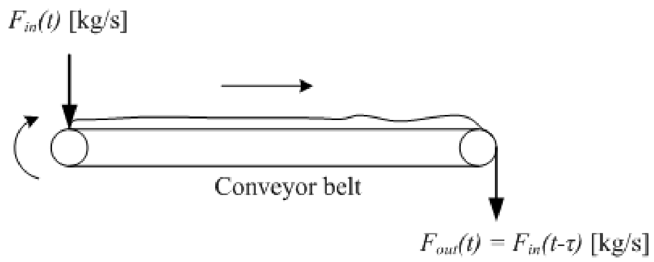

Figure 6.3: Time-delay on a conveyor belt

What is the transfer function of a time-delay? Taking the Laplace transform of (6.30) using (4.16):

$$F_{out}(s) = \underbrace{e^{-\tau s}}_{H(s)} F_{in}(s) \tag{6.31}$$

Thus, the transfer function of a time-delay of τ [time unit] is

$$H(s) = e^{-\tau s} \tag{6.32}$$

Figure 6.4 shows a simulation of a time-delay. The time-delay is $\tau = 1$ sec.

6.5 Higher order systems

Systems having higher order of the denominator polynomial of the transfer function than one, are so-called higher order systems, or more specifically, second order systems, third order systems and so on. A serial connection of first order systems results in a higher order system. (But not all possible higher order systems can be constructed by serial connection of first order systems.) When transfer functions are connected in series, the resulting

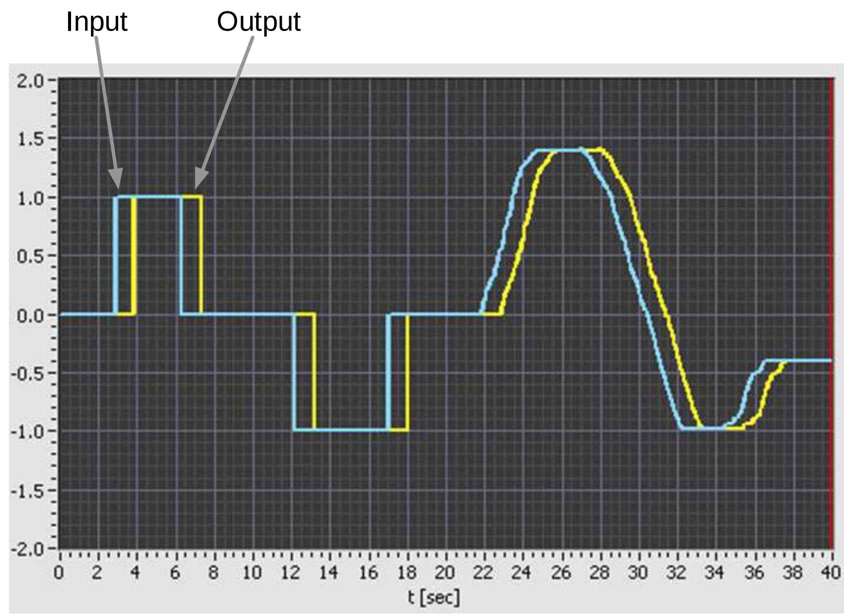

Figure 6.4: Output is equal to input, time delayed 1 sec.

transfer function is the product of the individual transfer functions, cf. Figure 5.2. As an example, Figure 6.5 shows a second order system consisting of “two time-constants” connected in series. The combined transfer function becomes

$$H(s) = \frac{1}{(T_1 s + 1)(T_2 s + 1)} = \frac{y_2(s)}{u(s)} \tag{6.33}$$

The figure also shows the step responses in the system. It is assumed that $T_1 = 1$, $T_2 = 1$ and $K = 1$. Observe that each first order systems makes the response become more sluggish, as it has a smoothing effect.

Let us define the *response-time* T_r as the *time it takes for a step response to reach 63% of its steady-state value*. For time-constant systems, the response-time is equal to the time-constant:

$$T_r = T \tag{6.34}$$

For higher order systems (order larger than one) it turns out that the response-time can be roughly estimated as the sum of the time-constants of the assumed serial subsystems that make up the higher order system:

$$T_r \approx \sum_i T_i \tag{6.35}$$

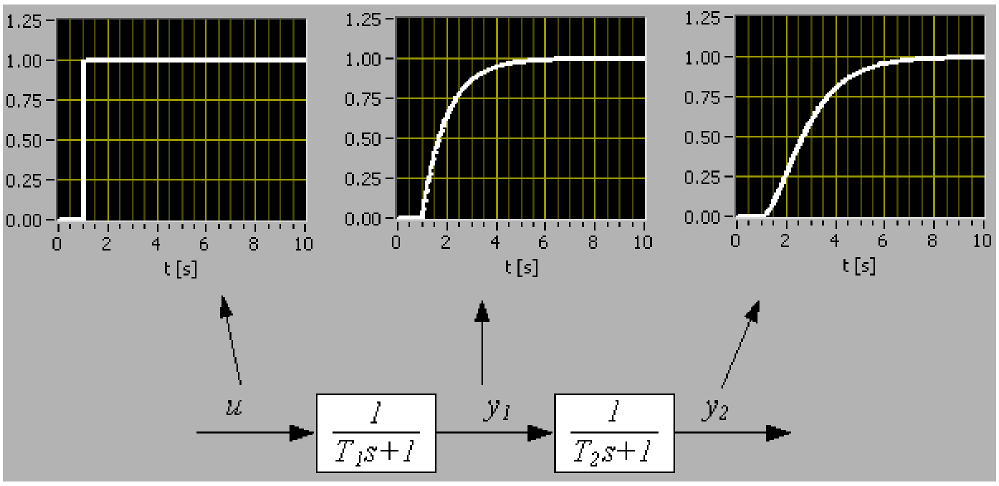

Figure 6.5: Step responses in a second order system

As an example, the response-time of the system shown in Figure 6.5 is

$$T_r \approx 1 + 1 = 2 \text{ s} \tag{6.36}$$

Does the simulation shown in Figure 6.5 confirm this?[2]

[2] Yes