Exercises to Basic Dynamics and Control

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Chapter 4

The Laplace transform

Exercise 4.1

Calculate the Laplace transform, $F(s)$, of the time function

$$f(t) = e^{-t} \tag{4.1}$$

using the definition of the Laplace transform.

Can you find the same answer ($F(s)$) by using a proper Laplace transform pair?

Exercise 4.2

Given the following differential equation:

$$\dot{y}(t) = -2y(t) + u(t) \tag{4.2}$$

with initial value $y(0) = 4$. Assume that the input variable $u(t)$ is a step of amplitude 1 at time $t = 0$.

1. Calculate the response in the output variable, $y(t)$, using the Laplace transform.
2. Calculate the steady-state value of $y(t)$ using the Final Value Theorem. Also calculate the steady-state value, y_s, from $y(t)$, and from (4.2) directly. Are all these values of y_s the same?

Chapter 5

Transfer functions

5.1 Introduction

No exercises here.

5.2 Definition of the transfer function

Exercise 5.1

In Exercise 3.2 the mathematical model of a wood-chip tank was derived. The model is

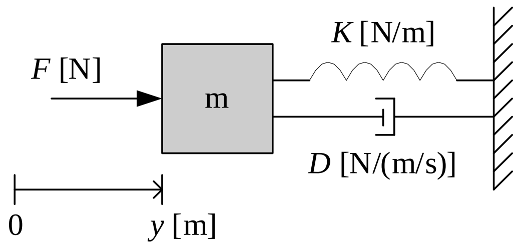
$$\rho A\dot{h}(t) = K_s u(t-\tau) - w_{out}(t) \tag{5.1}$$

Calculate the transfer function $H_1(s)$ from the screw control signal u to the level h and the transfer function $H_2(s)$ from the outflow w_{out} to the level h. (Tip: Use Eq. (4.16) in the text-book to calculate the Laplace transform of the time-delay.)

Exercise 5.2

Figure 5.2 shows a mass-spring-damper-system. y is position. F is applied force. D is damping constant. K is spring constant. It is assumed that the damping force F_d is proportional to the velocity, and that the spring force F_s is proportional to the position of the mass. The spring force is assumed to be zero when y is zero. Force balance (Newtons 2. Law) yields

$$m\ddot{y}(t) = F(t) - D\dot{y}(t) - Ky(t) \tag{5.2}$$



Calculate the transfer function from force F to position y.

5.3 Characteristics of transfer functions

Exercise 5.3

Given the following transfer function:

$$H(s) = \frac{s+3}{s^2+3s+2} \tag{5.3}$$

1. What is the order?
2. What is the characteristic equation?
3. What is the characteristic polynomial?
4. What are the poles and the zeros?

5.4 Combining transfer functions blocks in block diagrams

Exercise 5.4

Given a thermal process with transfer function $H_p(s)$ from supplied power P to temperature T as follows:

$$T(s) = \underbrace{\frac{b_p}{s+a_p}}_{H_p(s)} P(s) \tag{5.4}$$

The transfer function from temperature T to temperature measurement T_m is as follows:

$$T_m(s) = \underbrace{\frac{b_m}{s + a_m}}_{H_m(s)} T(s) \tag{5.5}$$

a_p, b_p, a_m, and b_m are parameters.

1. Draw a transfer function block diagram of the system (process with sensor) with P as input variable and T_m as output variable.
2. What is the transfer function from P to T_m? (Derive it from the block diagram.)

5.5 How to calculate responses from transfer function models

Exercise 5.5

Given the transfer function model

$$y(s) = \underbrace{\frac{5}{s}}_{H(s)} u(s) \tag{5.6}$$

Suppose that the input u is a step from 0 to 3 at $t = 0$. Calculate the response $y(t)$ due to this input.

5.6 Static transfer function and static response

Exercise 5.6

See Exercise 5.2. It can be shown that the transfer function from force F to position y is

$$H(s) = \frac{y(s)}{F(s)} = \frac{1}{ms^2 + Ds + K} \tag{5.7}$$

Calculate the static transfer function H_s. From H_s calculate the static response y_s corresponding to a constant force, F_s.

Chapter 6

Dynamic characteristics

6.1 Introduction

No exercises here.

6.2 Integrators

Exercise 6.1

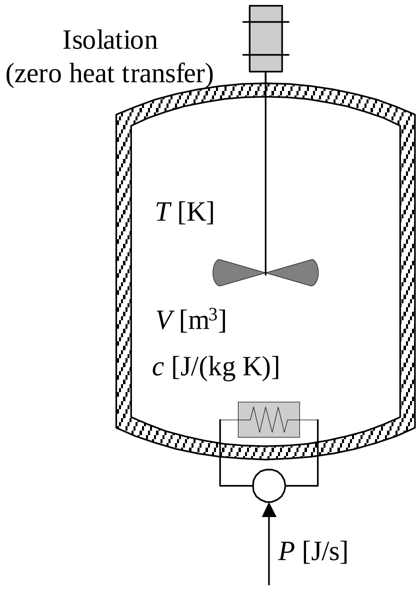
See Exercise 5.1. The transfer function from w_{out} to h is

$$\frac{h(s)}{w_{out}(s)} = -\frac{1}{\rho A s} = H_2(s) \tag{6.1}$$

1. Does this transfer function represent integrator dynamics?
2. Assume that $w_{out}(t)$ is a step from 0 to W at time $t = 0$. Calculate the response $h(t)$ that this excitation causes in the level h. You are required to base your calculations on the Laplace transform.

Exercise 6.2

Figure 6.2 shows an isolated tank (having zero heat transfer through the walls). Show that the tank dynamically is an integrator with the power P as input variable and the temperature T as output variable. (Hint: Study the transfer function from P to T.)



6.3 Time-constants

Exercise 6.3

Calculate the gain and the time-constant of the transfer function

$$H(s) = \frac{y(s)}{u(s)} = \frac{2}{4s+8} \tag{6.2}$$

and draw by hand roughly the step response of $y(t)$ due to a step of amplitude 6 in u from the following information:

- The steady-state value of the step response
- The time-constant
- The initial slope of the step response, which is

$$S_0 = \dot{y}(0^+) = \frac{KU}{T} \tag{6.3}$$

  (This can be calculated from the differential equation (6.12) in the text-book by setting $y(0) = 0$.)

Exercise 6.4

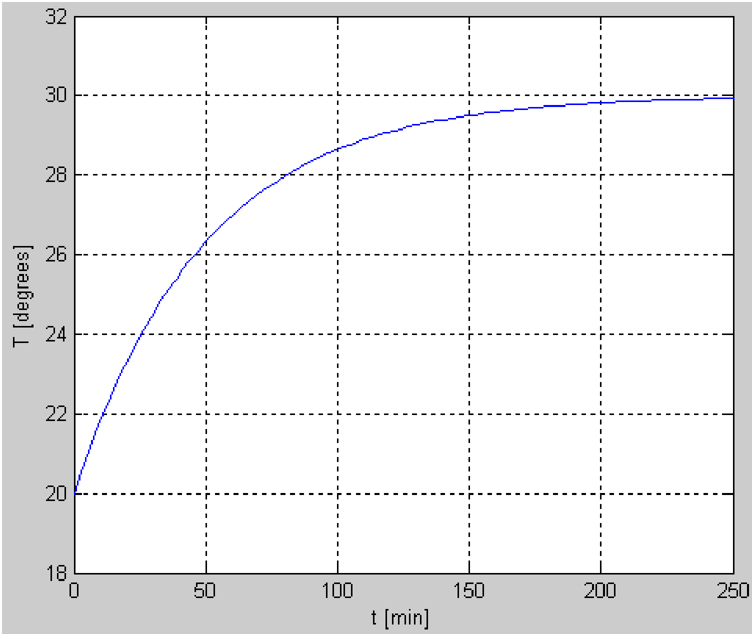


Figure 6.1:

Figure 6.1 shows the temperature response T of a thermal system due to a step of amplitude 1 kW in the supplied power P. Find the transfer function from ΔP (power) to ΔT (temperature) where Δ indicates deviations from the steady-state values. Assume that the system is of first order (a time-constant system).

Exercise 6.5

Figure 6.2 shows an RC-circuit (the circuit contains the resistor R and the capacitor C). The RC-circuit is frequently used as an analog lowpass filter:

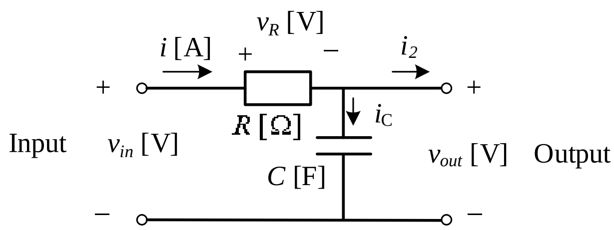


Figure 6.2: RC-circuit

Signals of *low* frequencies *passes* approximately unchanged through the filter, while signals of high frequencies are approximately filtered out (stopped). It can be shown that a mathematical model of the RC circuit is

$$RC\dot{v}_{out} = v_{in} - v_{out} \tag{6.4}$$

1. Calculate the transfer function $H(s)$ from v_{in} to v_{out}, and calculate the gain and the time-constant of $H(s)$.
2. Assume that the RC circuit is used as a signal filter. Assume that the capacitance C [F] is fixed. How can you adjust the resistance R (increase or descrease) so that the filter performs stronger filtering or, in other words: is more sluggish.

6.4 Time-delays

Exercise 6.6

For a pipeline of length 0.5 m and cross sectional area of 0.01 m^2 filled with liquid which flows with a volumetric flow 0.001 m^3/s, calculate the time-delay (transport delay) from inlet to outlet of the pipe.

6.5 Higher order systems

Exercise 6.7

Assume that a system can be well described by 3 time-constant systems in series, with the following time-constants respectively: 0.5, 1, and 2 sec. What is the approximate response time of the system?

Part II

SOLUTIONS

which can be written as

$$\underline{\underline{RC\dot{v}_2 - v_2 = RC\dot{v}_1}} \tag{12.31}$$

Solution to Exercise 4.1

We set $f(t) = e^{-t}$ in the integral that defines the Laplace transform:

$$\begin{aligned}
\underline{\underline{\mathcal{L}\{e^{-t}\}}} &= \int_0^\infty e^{-st}e^{-t}dt \\
&= \int_0^\infty e^{-(s+1)t}dt \\
&= \frac{1}{-(s+1)}\left[e^{-(s+1)t}\right]_{t=0}^{t=\infty} \\
&= \frac{1}{-(s+1)}[0-1] \\
&= \underline{\underline{\frac{1}{s+1}}}
\end{aligned}$$

The proper Laplace transform pair is:

$$\frac{k}{Ts+1} \quad \Longleftrightarrow \quad \frac{ke^{-t/T}}{T} = e^{-t} \tag{12.32}$$

Here, $T = 1$ and $k = 1$. Thus, $F(s)$ becomes

$$\underline{\underline{F(s) = \frac{1}{s+1} = \mathcal{L}\{e^{-t}\}}} \tag{12.33}$$

which is the same as found above using the definition of the Laplace transform.

Solution to Exercise 4.2

1. To calculate $y(t)$ we start by taking the Laplace transform of both sides of the given differential equation:

   $$\mathcal{L}\left\{\dot{y}(t)\right\} = \mathcal{L}\left\{-2y(t) + u(t)\right\} \tag{12.34}$$

   Here, we apply the time derivative property, cf. Eq. (4.10) in the text-book, at the left side, and the linear combination property, cf. Eq. (4.14) in the text-book, to the right side, to get

   $$sY(s) - 4 = -2Y(s) + U(s) \tag{12.35}$$

Here,

$$U(s) = \frac{1}{s} \tag{12.36}$$

since the Laplace transform of a step of amplitude 1 is $\frac{1}{s}$, cf. transform pair (4.7) in the text-book.

By now we have

$$sY(s) - 4 = -2Y(s) + \frac{1}{s} \tag{12.37}$$

Solving for $Y(s)$ gives

$$Y(s) = \underbrace{\frac{4}{s+2}}_{Y_1(s)} + \underbrace{\frac{1}{(s+2)\,s}}_{Y_2(s)} \tag{12.38}$$

To get the corresponding $y(t)$ from this $Y(s)$ we take the inverse Laplace transform of $Y_1(s)$ and $Y_2(s)$ to get $y_1(t)$ and $y_2(t)$ respectively, and then we calculate $y(t)$ as

$$y(t) = y_1(t) + y_2(t) \tag{12.39}$$

according to the linearity property of the Laplace transform. $y_1(t)$ and $y_2(t)$ are calculated below.

Calculation of $y_1(t)$:

We can use the transform pair (4.10) in the text-book, which is repeated here:

$$\frac{k}{Ts+1} \iff \frac{ke^{-t/T}}{T} \tag{12.40}$$

We have

$$Y_1(s) = \frac{4}{s+2} = \frac{2}{0.5s+1} \tag{12.41}$$

Hence, $k = 2$, and $T = 0.5$. Therefore,

$$y_1(t) = \frac{ke^{-t/T}}{T} = \frac{2e^{-t/0.5}}{0.5} = 4e^{-2t} \tag{12.42}$$

Calculation of $y_2(t)$:

We can use the transform pair (4.11) in the text-book, which is repeated here:

$$\frac{k}{(Ts+1)s} \iff k\left(1 - e^{-t/T}\right) \tag{12.43}$$

We have

$$Y_2(s) = \frac{1}{(s+2)\,s} = \frac{0.5}{(0.5s+1)\,s} \tag{12.44}$$

Hence, $k = 0.5$, and $T = 0.5$. Therefore,

$$y_2(t) = k\left(1 - e^{-t/T}\right) = 0.5\left(1 - e^{-t/0.5}\right) = 0.5\left(1 - e^{-2t}\right) \quad (12.45)$$

The final result becomes

$$\underline{\underline{y(t)}} = y_1(t) + y_2(t) \quad (12.46)$$

$$= 4e^{-2t} + 0.5\left(1 - e^{-2t}\right) \quad (12.47)$$

$$= \underline{\underline{0.5 + 3.5e^{-2t}}} \quad (12.48)$$

2. Using the Final Value Theorem on (12.38):

$$\underline{\underline{y_s}} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\left[\frac{4}{s+2} + \frac{1}{(s+2)\,s}\right] \quad (12.49)$$

$$= \lim_{s\to 0} s\frac{4}{s+2} + \lim_{s\to 0} s\frac{1}{(s+2)\,s} = 0 + \frac{1}{2} = \underline{\underline{0.5}} \quad (12.50)$$

From (12.48) we get

$$y_s = \lim_{t\to\infty} y(t) = 0.5 \quad (12.51)$$

And from the differential equation we get (because the time-derivative is zero in steady-state)

$$0 = -2y_s(t) + u_s(t) \quad (12.52)$$

which gives

$$y_s = \frac{u_s}{2} = \frac{1}{2} = 0.5 \quad (12.53)$$

So, the three results are the same.

Solution to Exercise 5.1

The Laplace transform of (5.1) is

$$\rho A\left[sh(s) - h_0\right] = K_s e^{-\tau s} u(s) - w_{out}(s) \quad (12.54)$$

Solving for output variable h gives

$$h(s) = \frac{1}{s}h_0 + \underbrace{\frac{K_s}{\rho A s}e^{-\tau s}}_{H_1(s)}u(s) + \underbrace{\left(-\frac{1}{\rho A s}\right)}_{H_2(s)}w_{out}(s) \quad (12.55)$$

Thus, the transfer functions are

$$\underline{\underline{H_1(s) = \frac{K_s}{\rho A s}e^{-\tau s}}} \quad (12.56)$$

and

$$\underline{\underline{H_2(s) = -\frac{1}{\rho A s}}} \tag{12.57}$$

Solution to Exercise 5.2

Laplace transform of (5.2) gives

$$m\left[s^2 y(s) - s\dot{y}_0 - y_0\right] = F(s) - D\left[sy(s) - y_0\right] - Ky(s) \tag{12.58}$$

Setting initial values $y_0 = 0$ and $\dot{y}_0 = 0$, and then solving for $y(s)$ gives

$$y(s) = \underbrace{\frac{1}{ms^2 + Ds + K}}_{H(s)} F(s) \tag{12.59}$$

The transfer function is

$$\underline{\underline{H(s) = \frac{y(s)}{F(s)} = \frac{1}{ms^2 + Ds + K}}} \tag{12.60}$$

Solution to Exercise 5.3

1. Order: $\underline{\underline{2}}$.
2. $\underline{\underline{s^2 + 3s + 2 = 0}}$
3. $\underline{\underline{s^2 + 3s + 2}}$
4. We write the transfer function on pole-zero-form:

$$H(s) = \frac{s+3}{s^2 + 3s + 2} = \frac{s+3}{(s+1)(s+2)} \tag{12.61}$$

   We see that the $\underline{\underline{\text{poles are } -1 \text{ and } -2}}$, and the $\underline{\underline{\text{zero is } -3}}$.

Solution to Exercise 5.4

1. Figure 12.3 shows the block diagram.
2. According to the *series combination rule* the transfer function becomes

$$\underline{\underline{H(s) = \frac{T_m(s)}{P(s)} = H_m(s)H_p(s) = \frac{b_m}{s + a_m}\frac{b_p}{s + a_p}}} \tag{12.62}$$

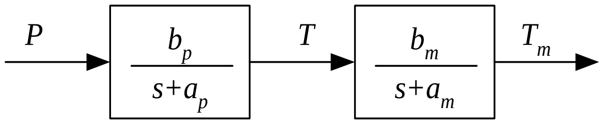


Figure 12.3:

Solution to Exercise 5.5

The Laplace transform of $u(t)$ is (cf. Eq. (4.7) in the text-book)

$$u(s) = \frac{3}{s} \tag{12.63}$$

Inserting this into (5.6) gives

$$y(s) = \frac{5}{s} \cdot \frac{3}{s} = \frac{15}{s^2} \tag{12.64}$$

which has the same form as in the Laplace transform pair given by Eq. (4.8) in the text-book. This transform pair is repeated here:

$$\frac{k}{s^2} \iff kt \tag{12.65}$$

We have $k = 15$, so the response is

$$\underline{\underline{y(t) = 15t}} \tag{12.66}$$

Solution to Exercise 5.6

Setting $s = 0$ in the transfer function gives

$$\underline{\underline{H_s}} = H(0) = \underline{\underline{\frac{1}{K}}} \tag{12.67}$$

The static response y_s corresponding to a constant force, F_s, is

$$\underline{\underline{y_s}} = H_s F_s = \underline{\underline{\frac{F_s}{K}}} \tag{12.68}$$

Solution to Exercise 6.1

1. Yes! Because the transfer function has the form of K_i/s.

2. The Laplace transform of the response is

$$h(s) = H_2(s)w_{out}(s) = -\frac{1}{\rho A s}w_{out}(s) \tag{12.69}$$

Since $w_{out}(t)$ is a step of amplitude W at $t = 0$, $w_{out}(s)$ becomes (cf. Eq. (4.7) in the text-book)

$$w_{out}(s) = \frac{W}{s} \tag{12.70}$$

With this $w_{out}(s)$ (12.69) becomes

$$h(s) = -\frac{1}{\rho A s}\frac{W}{s} \tag{12.71}$$

According to Eq. (4.8) in the text-book),

$$h(t) = -\frac{W}{\rho A}t \tag{12.72}$$

That is, the response is a ramp with negative slope.

Comment: This $h(t)$ is only the *contribution* from the outflow to the level. To calculate the complete response in the level, the total model (5.1), where both u and w_{out} are independent or input variables, must be used.

Solution to Exercise 6.2

Energy balance:

$$c\rho V\frac{dT}{dt} = P \tag{12.73}$$

Laplace transformation:

$$c\rho V\left[sT(s) - T_0\right] = P(s) \tag{12.74}$$

which yields

$$T(s) = \frac{1}{s}T_0 + \underbrace{\frac{1}{c\rho V s}}_{H(s)}P(s) \tag{12.75}$$

The transfer function is

$$H(s) = \frac{T(s)}{P(s)} = \frac{1}{c\rho V s} = \frac{K}{s} \tag{12.76}$$

which is the transfer function of an integrator with gain $K = 1/c\rho V$.

Solution to Exercise 6.3

We manipulate the transfer function so that the constant term of the denominator is 1:

$$H(s) = \frac{2}{4s+8} = \frac{2/8}{(4/8)\,s + 8/8} = \frac{0.25}{0.5s+1} = \frac{K}{Ts+1} \tag{12.77}$$

Hence,

$$\underline{\underline{K = 0.25; \;\; T = 0.5}} \tag{12.78}$$

We base the drawing of the step response on the following information:

- The steady-state value of the step response:

$$y_s = KU = 0.25 \cdot 6 = 1.5 \tag{12.79}$$

- The time-constant:

$$T = 0.5 \tag{12.80}$$

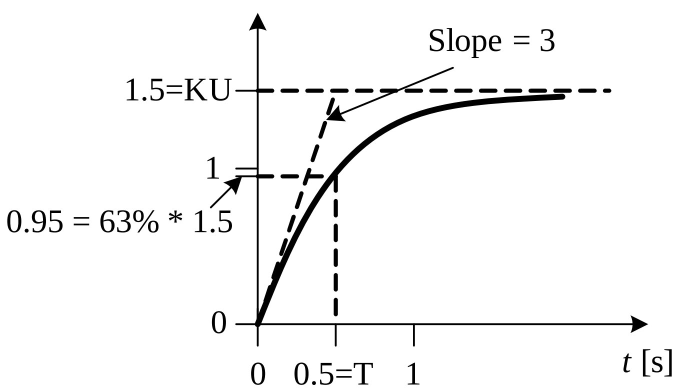
  which is the time when the step response has reached value

$$0.63 \cdot y_s = 0.63 \cdot 1.5 = 0.95 \tag{12.81}$$

- The initial slope of the step response:

$$S_0 = \dot{y}(0^+) = \frac{KU}{T} = \frac{0.25 \cdot 6}{0.5} = 3 \tag{12.82}$$

Figure II shows the step response.



Solution to Exercise 6.4

From Figure 6.1 we see that the gain is

$$K = \frac{\Delta T}{\Delta P} = \frac{30 \text{ K} - 20 \text{ K}}{1 \text{ kW}} = 10 \frac{\text{K}}{\text{kW}} \tag{12.83}$$

and that the time constant (the 63% rise time) is

$$T_1 = 50 \text{ min} \tag{12.84}$$

The transfer function becomes

$$\underline{\underline{\frac{\Delta T(s)}{\Delta P(s)} = \frac{10}{50s+1} \frac{\text{K}}{\text{kW}}}} \tag{12.85}$$

Solution to Exercise 6.5

1. Laplace transformation of the differential equation (6.4) gives

$$RCsv_{out}(s) = v_{in}(s) - v_{out}(s) \tag{12.86}$$

   Solving for $v_{out}(s)$ gives

$$v_{out}(s) = \frac{1}{RCs+1} v_{in}(s) \tag{12.87}$$

   The transfer function is

$$\underline{\underline{H(s) = \frac{1}{RCs+1}}} = \frac{K}{Ts+1} \tag{12.88}$$

   The gain is

$$\underline{\underline{K = 1}} \tag{12.89}$$

   The time-constant is

$$\underline{\underline{T = RC}} \tag{12.90}$$

2. The filtering is stronger if $\underline{\underline{R \text{ is increased}}}$.

Solution to Exercise 6.6

The time-delay is

$$\underline{\underline{\tau}} = \frac{AL}{q} = \frac{0.01 \text{ m}^2 \cdot 0.5 \text{ m}}{0.001 \text{ m}^3/\text{s}} = \underline{\underline{5 \text{ s}}} \tag{12.91}$$

Solution to Exercise 6.7

The approximate response time is

$$\underline{\underline{T = 0.5 + 1 + 2 = 3.5 \text{ s}}} \tag{12.92}$$

Solution to Exercise 7.1

With $T_f = 2$ sec the filter will be much more sluggish than the motor. Quick motor speed changes will then be filtered or smoothed out (depending on how quick the real speed actually varies).

A good estimate of the filter time-constant is one tenth of the process time-constant:

$$\underline{\underline{T_f}} = \frac{0.2\text{s}}{10} = \underline{\underline{0.02 \text{ s}}} \tag{12.93}$$

Solution to Exercise 7.2

The slope a can be calculated from

$$\underline{\underline{a}} = \frac{T_{\max} - T_{\min}}{M_{\max} - M_{\min}} = \frac{55 - 15}{20 - 4} = \frac{40}{16} = \underline{\underline{2.5 \, \frac{^{\text{o}}\text{C}}{\text{mA}}}} \tag{12.94}$$

and

$$\underline{\underline{b}} = T_{\min} - aM_{\min} = 15 \, ^{\text{o}}\text{C} - 2.5 \, \frac{^{\text{o}}\text{C}}{\text{mA}} \cdot 4 \text{ mA} = \underline{\underline{5 \, ^{\text{o}}\text{C}}} \tag{12.95}$$

Solution to Exercise 7.3

The slope a can be calculated from

$$\underline{\underline{a}} = \frac{u_{1\max} - u_{1\min}}{u_{\max} - u_{\min}} = \frac{20 - 4}{3336 - 0} = \underline{\underline{\frac{16}{3336} \, \frac{\text{mA}}{\text{kg/min}} = 0.0048 \, \frac{\text{mA}}{\text{kg/min}}}} \tag{12.96}$$

and

$$\underline{\underline{b}} = u_{1\min} - au_{\min} = 4 \text{ mA} - \frac{16}{3336} \, \frac{\text{mA}}{\text{kg/min}} \cdot 0 \, \frac{\text{kg}}{\text{min}} = \underline{\underline{4 \text{ mA}}} \tag{12.97}$$

The scaling function $u_1 = au + b$ is used to transform the flow value in kg/min demanded by the level controller (as the controller output signal) to a corresponding currect signal in mA to be applied to the feed screw.

Solution 7.4