5.4.5 Computer Tools for Calculating Frequency Responses from Transfer Functions

Calculating the frequency response and drawing it in a Bode diagram can be quite time-consuming, so you should use some computer tool for doing it. A document available from the webpage of this book at http://techteach.no describes such tools in MATLAB and LabVIEW. For example, the `bode`-function in Control System Toolbox in MATLAB does the job.

5.5 Application of Frequency Response: Signal Filters

5.5.1 Introduction

A *signal filter* – or just *filter* – is used to attenuate (ideally: remove) a certain frequency interval of frequency components from a signal. These frequency components are typically noise. For example, a lowpass filter is used to attenuate high-frequent components (low-frequent components passes).

Knowledge about filtering functions is crucial in signal processing, but it is useful also in control engineering because control systems can be regarded as filters in the sense that the controlled process variable can follow only a certain range or interval of frequency components in the reference (setpoint) signal, and it will be only a certain frequency range of process disturbances that the control system can compensate for effectively. Furthermore, knowledge about filters can be useful in the analysis and design of physical processes. For example, a stirred tank in a process line can act as a lowpass filter since it attentuates low-frequent components in the inflow to the tank.

In this section we will particularly study *lowpass filters*, which is the most commonly used filtering function, but we will also take a look at *highpass filters*, *bandpass filters* and *bandstop filters*.

Figure 5.8 shows the gain function for ideal filtering functions and for practical filters (the phase lag functions are not shown). The *passband* is the frequency interval where the gain function has value 1, ideally (thus, frequency components in this frequency interval passes through the filter, unchanged). The *stopband* is the frequency interval where the gain

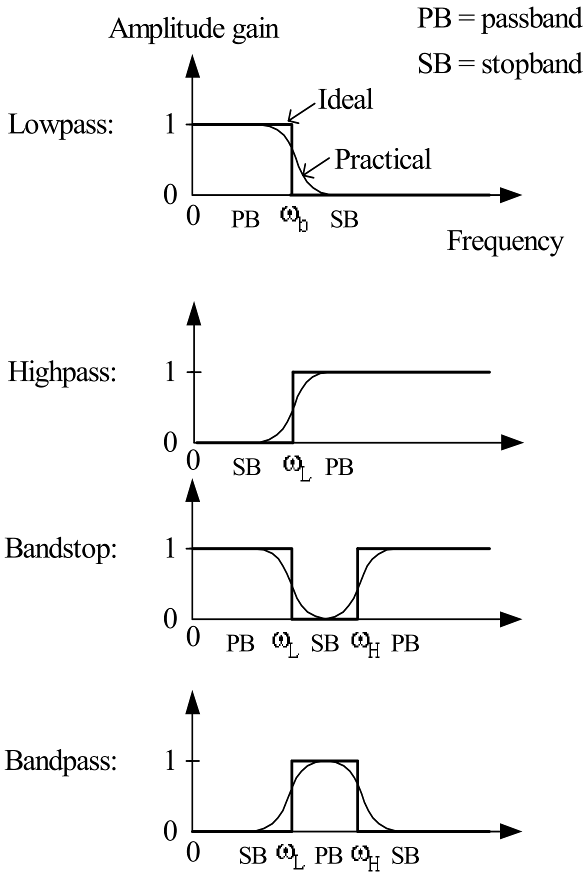

Figure 5.8: The gain functions for ideal filters and for practical filters of various types.

function has value 0, ideally (thus, frequency components in this frequency interval are stopped through the filter).[5]

It can be shown that transfer functions for ideal filtering functions will have infinitely large order. Therefore, ideal filters can not be realized, neither with analog electronics nor with a filtering algorithm in a computer program.

[5]It is a pity that lowpass filters were not called highstop filters in stead since the main purpose of a lowpass filter is to stop high-frequency components. Similarly, highpass filters should have been called lowstop filters, but it is too late now...

Tools for filter design

MATLAB (Signal Processing Toolbox), Octave, and LabVIEW have many functions for design of signal filters. Information about these tools is available from the home page of this book at http://techteach.no.

5.5.2 Lowpass Filters

First order lowpass filters

The transfer function of a first order lowpass filter with input variable u and output variable y is usually written on the

$$H(s) = \frac{1}{\frac{s}{\omega_b} + 1} \tag{5.63}$$

where ω_b [rad/s] is the *bandwidth* of the filter. This is a first order transfer function with gain $K = 1$ and time-constant $T = 1/\omega_b$, cf. (4.15). The frequency response is

$$\begin{aligned} H(j\omega) &= \frac{1}{\frac{j\omega}{\omega_b} + 1} & (5.64) \\ &= \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}\, e^{j \arctan \frac{\omega}{\omega_b}}} \\ &= \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}}\, e^{j\left(-\arctan \frac{\omega}{\omega_b}\right)} & (5.65) \end{aligned}$$

The gain function is

$$|H(j\omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}} \tag{5.66}$$

and the phase lag function is

$$\arg H(j\omega) = -\arctan \frac{\omega}{\omega_b} \tag{5.67}$$

Figure 5.4 on page 5.4 shows exact and asymptotic curves of $|H(j\omega)|$ and $\arg H(j\omega)$ drawn in a Bode diagram. In the figure, $K = 1$ and $\omega_b = \omega_c$.

The bandwidth defines the upper limit of the passband. It is common to say that the bandwidth is the frequency where the filter gain is

$1/\sqrt{2} = 0.71 \approx -3$ dB (above the bandwidth the gain is less than $1/\sqrt{2}$). This bandwidth is therefore referred to as the "−3 dB-bandwidth". Now, what is the −3 dB-bandwidth of a first order lowpass filter? It is the ω-solution of the equation

$$|H(j\omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}} = \frac{1}{\sqrt{2}} \tag{5.68}$$

The solution is $\omega = \omega_b$. Therefore, ω_b [rad/s] given in (5.63) is the −3 dB-bandwidth in rad/s. In Hertz the bandwidth is

$$f_b = \frac{\omega_b}{2\pi} \tag{5.69}$$

What is the *response-time* T_r (63% rise time of the step response) of a first order lowpass filter? For first order systems T_r equals the time-constant T. Thus,

$$T_r = T = \frac{1}{\omega_b} \tag{5.70}$$

Figure 5.9 shows the front panel of a simulator of a first order filter where the input signal consists of a sum of two sinusoids or frequency components of frequency less than and greater than, respectively, the bandwidth. The simulation shows that the low frequent component (0.5 Hz) passes almost unchanged (it is in the passband of the filter), while the high-frequent component (8 Hz) is attenuated (it lies in the stopband).

Example 42 ***The RC-circuit as a lowpass filter***

In Example 6 on page 34 we found the following model of an RC-circuit:

$$RC\dot{v}_2(t) = v_1(t) - v_2(t) \tag{5.71}$$

The transfer function from the input voltage v_1 to the output voltage v_2 becomes

$$H_{v_2,v_1}(s) = \frac{1}{RCs + 1} = \frac{1}{\frac{s}{\omega_b} + 1} \tag{5.72}$$

Thus, the RC-circuit is a first order lowpass filter with bandwidth

$$\omega_b = \frac{1}{RC} \text{ rad/s} \tag{5.73}$$

If for example $R = 1\text{k}\Omega$ and $C = 10\mu\text{F}$, the bandwidth is $\omega_b = 1/RC = 100$ rad/s. (5.73) can be used to design the RC-circuit (calculate the R- and C-values).

[End of Example 42]

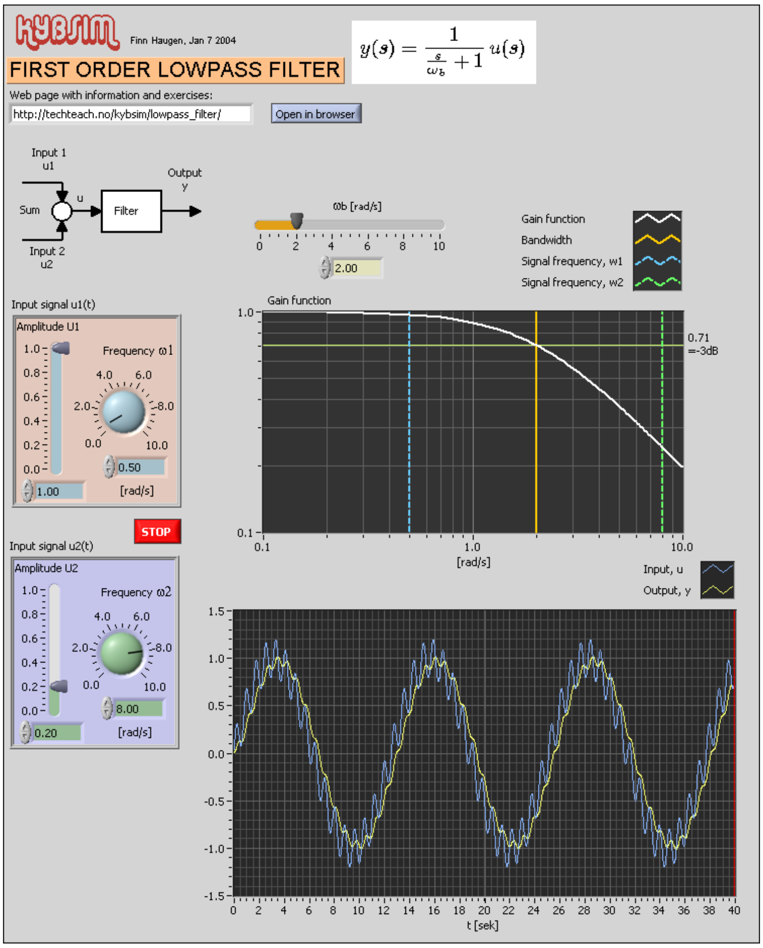

Figure 5.9: Simulator for a first order lowpass filter where the input signal consists of a sum of two frequency componens

Example 43 ***Liquid tank as a lowpass filter***

In Example 3 on page 23 we developed the following model of a heated liquid tank:

$$c\rho V\dot{T} = P + cw\left(T_i - T\right) \tag{5.74}$$

After taking the Laplace transform of the differential equation we can find the following transfer function from the inlet temperature T_i to the tank temperature T:

$$H_{T,T_i}(s) = \frac{1}{\frac{\rho V}{w}s + 1} = \frac{1}{\frac{s}{\omega_b} + 1} \tag{5.75}$$

Thus, the tank is a first order lowpass filter with bandwidth

$$\omega_b = \frac{w}{\rho V} \text{ [rad/s]} \tag{5.76}$$

The tank will attenuate temperature variations. (5.76) can be used to design the tank.

[End of Example 43]

Higher order lowpass filters

Above the filters (their transfer functions) were of order $n = 1$. However, the higher the order, the closer the gain function of the filter will be to the ideal gain function shown in Figure 5.8. The drawback with high filter order is an increased number of electronic components in an analog filter and a more complicated filtering algorithm in a programmed filter. Furthermore, it can be shown that the phase lag function becomes more negative the higher the order of the filter.

Higher order lowpass filters have an s-polynomial of order two or higher as a denominator. Depending on the filter topology which is chosen, the filter may have an s-polynomial of order zero or one or higher in the numerator.

A commonly used filter topology is *Butterworth* filters. Butterworth filters have the property that of all filters of same order (order of the s-polynomial in the denominator), Butterworth filters have the flattest gain curve in the passband. Butterworth filters have a constant (no s-plynomial) as the numerator in the transfer function. Other filter topologies are Chebyshev filters and Elliptic filters. Filters in these topologies have an s-polynomial in the numerator. These filters give a sharper corner of the passband, but there are ripples or peaks in the passband (Chebyshev Type I filters) or in the stopband (Chebyshev Type II filters) or in both (Elliptic filters). The magnitude of the ripples can be designed by the user.

Below are shown normalized Butterworth filters of order $n = 1, ..., 5$. "Normalized" means that the bandwidth is $\omega_b = 1$ rad/s. If you need a filter of bandwidth ω_b different from 1 rad/s, you substitute s in the proper transfer function with s/ω_b.

$$H_1(s) = \frac{1}{s+1} \tag{5.77}$$

$$H_2(s) = \frac{1}{s^2 + 1.414s + 1} \tag{5.78}$$

$$H_3(s) = \frac{1}{(s+1)(s^2+s+1)} \tag{5.79}$$

$$H_4(s) = \frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)} \tag{5.80}$$

$$H_5(s) = \frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)} \tag{5.81}$$

Figure 5.10 shows the gain functions of Butterworth filters above – all with same bandwidth of $\omega_b = 1$ rad/s. From the Bode plots we see that the

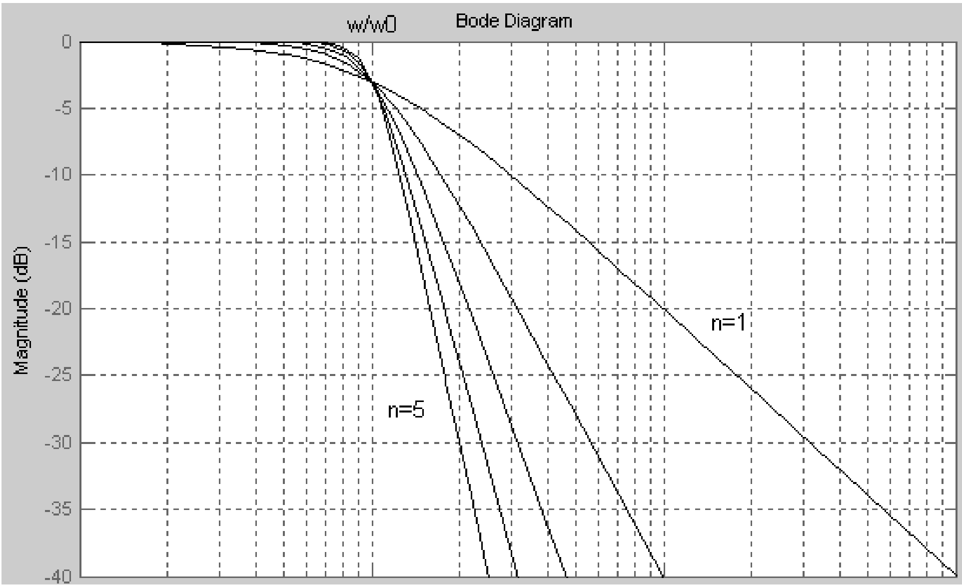

Figure 5.10: Bode plots of the gain functions of Butterworth filters of orders $1, ..., 5$ – all with bandwidth $\omega_b = 1$ rad/s.

higher order filter, the closer the gain function is to the ideal gain function (closer to 1 in the passband, and closer to $0 = -\infty$ dB in the stopband). It can be shown that the phase lag becomes more negative with increasing filter order, causing more time delay through the filter.

5.5.3 Developing Other Filters Using Frequency Transformation

You can use *frequency transformation* to develop transfer functions for lowpass filters, highpass filters, bandpass filters, and bandstop filters. In the procedure you always start with a known transfer function of a