SimView

Mathematical model of a pendulum on a cart

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5. July 2011

This document describes the mathematical model of the cart with pendulum and the LQ (Linear Quadratic) optimal control system used to stabilize the cart and the pendulum in the SimView[1] simulator LQ OPTIMAL CONTROL OF INVERTED PENDULUM.

1 Mathematical model

A reference for the system described here is the text-book Nonlinear Systems by H. K. Khalil (Pearson Education, 2000). The original reference is Linear Optimal Control Systems by H. Kwakernaak and R. Sivan (Wiley, 1972).

Figure 1 shows a pendulum mounted on a cart.

A motor (in the cart) acts on the cart with a force F. This force is manipulated by the controller to stabilize the pendulum in an upright position or in a downright position at a specified position of the cart.

A mathematical model of the system is derived below. This model is used to design a stabilizing controller, namely an optimal controller.The mathematical model is based on the following principles:

1. Force balance (Newton's Second Law) applied to the horizontal movement of the center of gravity of the pendulum:

$$m\frac{d^2}{dt^2}\left(y + L\sin a\right) = H \tag{1}$$

The differentiation must be carried out, but the result of it not shown here.

[1] http://techteach.no/simview

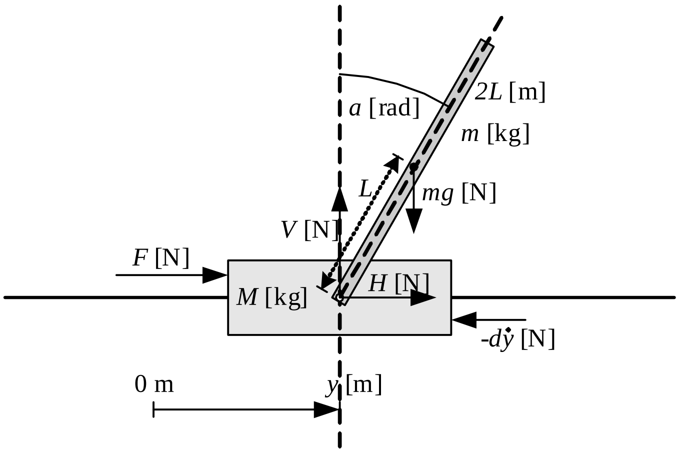

Figure 1:

2. Force balance applied to the vertical movement of the center of gravity of the pendulum:

$$m\frac{d^2}{dt^2}\left(L\cos a\right) = V - mg \tag{2}$$

The differentiation must be carried out, but the result of it not shown here.

3. Torque balance (the rotational version of the Newton's Second Law applied to the center of gravity of the pendulum:

$$I\ddot{a} = VL\sin a - HL\cos a \tag{3}$$

4. Force balance applied to the cart:

$$M\ddot{y} = F - H - d\dot{y} \tag{4}$$

In the above equations,

- I is the moment of inertia of the pendulum about it's center of gravity. For the pendulum shown in Figure 1,

$$I = \frac{mL^2}{12} \tag{5}$$

- V and H are vertical and horizontal forces, respectively, in the pivot.
- d is a damping coefficient.

From Eq. (1) – (4), the internal forces V and H can be eliminated, resulting in two differential equations (not containing V and H), which are not shown here. These two differential equations can be written as the following nonlinear state-space model:

$$\dot{x}_1 = x_2 \tag{6}$$

$$\dot{x}_2 = \frac{-m^2L^2g\cos x_3 \sin x_3 + \left(u + mL\, {x_4}^2 \sin x_3 - dx_2\right)\left(I + mL^2\right)}{D_1} \tag{7}$$

$$\dot{x}_3 = x_4 \tag{8}$$

$$\dot{x}_4 = \frac{(m+M)\,(mgL\sin x_3) - \left(u + mL\, {x_4}^2 \sin x_3 - dx_2\right) mL\cos x_3}{D_1} \tag{9}$$

where

$$D_1 = \left(I + mL^2\right)(m+M) - m^2L^2\cos^2 x_3 \tag{10}$$

In the above state-space model,

- $x_1 = y$ (cart horizontal position)
- $x_2 = \dot{y}$ (cart horizontal speed)
- $x_3 = a$ (pendulum angular position)
- $x_4 = \dot{a}$ (pendulum angular speed)

2 Controller

To stabilize the pendulum either vertically up or vertically down, a specified possibly non-zero position, r, of the cart, a steady-state LQ (Linear Quadratic Regulator) controller is used. The feedback control function is as follows:

$$u(t) = -G_{11}\left[x_1(t) - r(t)\right] - G_{12}x_2(t) - G_{13}x_3(t) - G_{14}x_4(t) \tag{11}$$

The controller output, u, is applied as the force F acting on the cart. Hence the force is calculated as a linear combination of the states of the system. The states are assumed to be available via measurements. (Thus, there is a feedback from the measured states to the process via the controller.)

The controller gains, G_{11}, G_{11}, G_{11}, G_{14}, are calculated by the LabVIEW MathScript function **lqr** which has the following syntax:

[G, X, E] = lqr(A, B, Q, R);

where G is the calculated gain. (X is the steady-state solution of the Riccatti equation, and E is the eigenvalue vector of the feedback closed loop system.) A and B are the matrices of the linear state-space process model corresponding to the nonlinear model (6) – (9). A and B are presented below. Q is the state variable weight matrix and R is the control variable weight matrix used in the LQR optimization criterion:

$$J = \int_{t=0}^{\infty} \Delta x^T(t) Q \Delta x(t) + \Delta u(t)^T R \Delta u(t) dt \tag{12}$$

Q has the following form:

$$Q = \begin{bmatrix} Q_{11} & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 \\ 0 & 0 & Q_{33} & 0 \\ 0 & 0 & 0 & Q_{44} \end{bmatrix} \tag{13}$$

while Q_{ii} can be used as controller tuning parameters. R is

$$R = [R_{11}] \tag{14}$$

and is also used as tuning parameter.

The control system can be used to stabilize the cart and the pendulum in two different operation points, namely vertically up and vertically down. The process model is nonlinear. Therefore, the linear matrices A and B are derived for each of these operating points. The linearization of (6) – (9) can be carried out using the following assumptions:

- When the angular position $x_3 = a$ is close to zero, $\cos x_3$ is approximately equal to 1.
- When the angular position $x_3 = a$ is close to $180° = \pi$ rad, $\cos x_3$ is approximately equal to -1.
- When the angular position $x_3 = a$ is close to zero or close to $180° = \pi$ rad, $\sin x_3$ is very similar to x_3, and hence is substituted by x_3.
- The angular speed $x_4 = \dot{a}$ is small, approximately zero in both operating points.

Using these assumptions, it can be shown that the linearized model becomes

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \\ \Delta\dot{x}_3 \\ \Delta\dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{\left(I+mL^2\right)d}{D_2} & -s\frac{m^2L^2g}{D_2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & s\frac{mLd}{D_2} & \frac{(m+M)mgL}{D_2} & 0 \end{bmatrix}}_{A} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{I+mL^2}{D_2} \\ 0 \\ -\frac{mL}{D_2} \end{bmatrix}}_{B} [\Delta u] \tag{15}$$

where

$$D_2 = \left(I + mL^2\right)(m + M) - m^2L^2 \tag{16}$$

and

$$s = \begin{cases} 1 \text{ for pendulum in vertical up position} \\ -1 \text{ for pendulum in vertical down position} \end{cases} \tag{17}$$

The total control signal is

$$u(t) = u_{\text{op}} + \Delta u(t) \tag{18}$$

where u_{op} is the control signal needed to keep the system at the operating point, nominally. However, $u_{\text{op}} = 0$ in both operating points.

Because the parameters of the linear model (parameters of A, only) are different in these two operating points, the controller gains will also be different.

3 Model for simulator and model for controller design

The model (1) – (4) is the basis of both the simulator (representing the real process) and the controller. However, to make it possible to check if the control system is robust, two sets of model parameters are available in the front panel of the simulator:

- Model parameters M_{real}, m_{real}, L_{real}, d_{real} used in the simulator.
- Model parameters M_{model}, m_{model}, L_{model}, d_{model} used in the design of the controller.

By default, these two parameter sets have equal numerical values, but if you want to check for robustness of the controller against variations or inaccurate knowledge about one certain parameter, you must set the values of the corresponding parameters to different values, e.g. set d_{model} to a different value from d_{real}.